SOME ASPECTS OF THE JACOBIAN CONJECTURE (THE GEOMETRY OF AUTOMORPHISMS OF $\mathbb{C}^2$)

A. Hamid A. Hussain Ali

A Thesis Submitted for the Degree of PhD
at the
University of St Andrews

1987

THE BRITISH LIBRARY DOCUMENT SUPPLY CENTRE
Boston Spa, Wetherby, West Yorkshire LS23 7BQ

ST. ANDREWS.

PhD Thesis by ALI A·H·A·H·

We have given the above thesis the Document Supply Centre identification number:

D 806 [illegible]

In your notification to Aslib please show this number, so that it can be included in their published <u>Index to Theses with Abstracts</u>.

J P CHILLAG
Theses Officer

SOME ASPECTS OF THE JACOBIAN CONJECTURE

(THE GEOMETRY OF AUTOMORPHISMS OF $\mathbb{C}^2$)

BY

A HAMID A HUSSAIN ALI

A thesis submitted for the degree of Doctor of Philosophy in the University of St Andrews.
St Andrews, January 1987.

Th A592

ABSTRACT

We consider the affine varieties which arise by considering invertible polynomial maps from $\mathbb{C}^2$ to itself of less than or equal to a given degree. These varieties arise naturally in the investigation of the long-standing Jacobian Conjecture.

We start with some calculations in the lower degree cases. These calculations provide a proof of the Jacobian conjecture in these cases and suggest how the investigation in the higher degree cases should proceed.

We then show how invertible polynomial maps can be decomposed as products of what we call triangular maps and we are able to prove a uniqueness result which gives a stronger version of Jung's theorem [J] which is one of the most important results in this area. Our proof also gives a new derivation of Jung's theorem from Segre's lemma.

We give a different decomposition of an invertible polynomial map as a composition of "irreducible maps" and we are able to write down standard forms for these irreducibles. We use these standard forms to give a description of the structure of the varieties of invertible maps.

We consider some interesting group actions on our varieties and show how these fit in with the structure we describe.

Abstract (contd)

Finally, we look at the problem of identifying polynomial maps of finite order. Our description of the structure of the above varieties allows us to solve this problem completely and we are able to show that the only elements of finite order are those which arise from conjugating linear elements of finite order.

DEDICATION

To my wife Khawla, my sons Khazraj, Hassan and Barra and my relatives in Iraq.

CERTIFICATE

I A Hamid A Hussain Ali hereby certify that this thesis has been composed by myself, that it is a record of my own work, and that it has not been accepted in partial or complete fulfilment of any other degree or professional qualification.

I was admitted to the Faculty of Science of the University of St Andrews under Ordinance General No 12 in October 1982 and as a candidate for the degree of PhD in February 1984.

Signed .... Date ..26/1/1987...

I hereby certify that the candidate has fulfilled the conditions of the Resolution and Regulations appropriate to the Degree of PhD.

Signed .... Date ..26/1/87.......

ACKNOWLEDGEMENTS

I take this opportunity to express my gratitude to Dr J J O'Connor, my supervisor for suggesting the project and for his kind advice, stimulating discussions and encouragement throughout my study period.

It is my pleasure to record sincere appreciation and gratitude to my wife, Khawla for her patience and encouragment throughout this study.

I owe a special thanks to the Iraqi Government for supporting me financially.

COPYRIGHT DECLARATION

CONTENTS

Chapter 1

INTRODUCTION

A mapping $f:\mathbb{C}^n \longrightarrow \mathbb{C}^n$, $f(\underline{x}) = (f_1(\underline{x}), \ldots\ldots, f_n(\underline{x}))$, is a polynomial mapping if each f_i is a polynomial.

If f is invertible (ie if f^{-1} is a polynomial map), then since the Jacobian satisfies $J(h \circ g) = Jh(g).Jg$, Jf is invertible also and hence (since the determinant of $J.f$ is a polynomial) det (Jf) must be a non-zero constant polynomial. The converse of this result was first conjectured by Keller [K] in 1939 and this is known as the Jacobian Conjecture.

Much evidence has been assembled in favour of this conjecture (see the article by Bass, Connell and Wright [BCW]), but even for the case $n = 2$, the conjecture is still open. (Though in this case Moh shows that it holds for all maps from $\mathbb{C}^2$ to $\mathbb{C}^2$ with degree less than or equal to 100 [Mo]).

If f is a general polynomial map of given degree, then the equation:

$$\det(Jf) = \text{Constant} \neq 0$$

can be regarded as a set of simultaneous equations in the coefficients of f and hence defines an affine variety in the space of all polynomial maps from $\mathbb{C}^n$ to itself. Since these varieties arise in such a natural way they are worthy of further study.

In this work, we study the case $n = 2$.

We first tackle the problem for maps of degree 2, 3 and 4 by solving the sets of simultaneous equations. This gives a proof of the Jacobian Conjecture for maps with these degrees and also allows us to describe standard forms for such maps. In the later chapters we show how similar standard forms can be found for maps of higher degree. This will enable us to give a detailed description of the structure of the varieties of such maps and such a description allows us to solve the problem of identifying polynomial maps of finite order.

Chapter 2

CALCULATIONS FOR INVERTIBLE POLYNOMIAL MAPS OF LOW DEGREE

In this chapter, the polynomial maps from $\mathbb{C}^2$ to itself of degrees 1, 2, 3 and 4 which map $(0,0) \in \mathbb{C}^2$ to itself are considered. (Later one can see proofs for some general results which avoid the necessity for some of these intricate calculations.)

Any polynomial map can be formed from the composition of an origin-preserving map with a translation map and from now on we assume that all maps are origin preserving.

The set of all origin-preserving linear maps

$$(x,y) \longmapsto (a_1\ x + a_2\ y,\ b_1\ x + b_2\ y)$$

forms a space of dimension 4 where this map is identified with the point:

$$(a_1, a_2, b_1, b_2) \in \mathbb{C}^4$$

(This identification is the standard Cremona map.)

For such maps the Jacobian Conjecture is true trivially and the set of all such invertible maps is an open subset of $\mathbb{C}^4$ given by:

$$\det Jf \neq 0 \quad \text{or } a_1 b_2 - a_2 b_1 \neq 0$$

and hence is an affine variety of dimension 4. So the first non-trivial case is the set of polynomial maps of degree ≤ 2

Theorem Any Polynomial map from $\mathbb{C}^2$ to itself of degree $\leq$ 2 whose Jacobian determinant is a non-zero constant is invertible and is of the form:

$$(x,y) \longmapsto ((sx + ty)^2 + a_4\ x + a_5\ y,\ k(sx + ty)^2 + b_4\ x + b_5\ y)$$

for some $s, t, a_4, a_5, b_4, b_5 \in \mathbb{C}$ where

$$\frac{s}{t} = \frac{b_4 - ka_4}{b_5 - ka_5} \quad \text{and } a_4 b_5 \neq a_5 b_4 .$$

The variety of all such maps is 6 dimensional.

<u>Proof</u> A typical origin-preserving polynomial map of degree ≤ 2 is of the form:

$$f(x,y) = (a_1 x^2 + a_2 xy + a_3 y^2 + a_4 x + a_5 y,$$

$$b_1 x^2 + b_2 xy + b_3 y^2 + b_4 x + b_5 y)$$

and so the set of all such maps forms a space of dimension 10.

The condition det Jf = constant $\neq 0$ is thus:

$$|Jf| = \begin{vmatrix} 2a_1 x + a_2 y + a_4 & a_2 x + 2a_3 y + a_5 \\ 2b_1 x + b_2 y + b_4 & b_2 x + 2b_3 y + b_5 \end{vmatrix} \neq 0$$

By equating coefficients of powers x and y, this is equivalent to the following set of 5 equations and one inequality:

x^2: $\quad a_1 b_2 = b_1 a_2$

xy: $\quad a_1 b_3 = b_1 a_3$

y^2: $\quad a_2 b_3 = b_2 a_3$

$$x: \quad 2a_1 b_5 + a_4 b_2 = a_2 b_4 + 2b_1 a_5$$

$$y: \quad 2b_3 a_4 + a_2 b_5 = 2b_4 a_3 + b_2 a_5$$

$$|Jf| = a_4 b_5 - a_5 b_4 = \text{constant} \neq 0$$

Note that these equations are not all independent. For example, the first two imply the third.

One can solve these equations to find a general form for a map of degree ≤ 2 with a constant determinant as follows:

One can assume $a_1 \neq 0$ (One of the 2nd order terms must be present. If, for example we have $a_1 = 0$ and $a_2 \neq 0$, then compose on the right with the map $(x,y) \longmapsto (x, \; ex + y)$ and we get a map with the required term for a suitable choice of e).

Let $b_1 = ka_1$, then $k = \dfrac{b_1}{a_1}$. We now equate coefficients of terms in the Jacobian determinant to zero.

Coeeficients of x^2: $\quad b_2 = \dfrac{b_1}{a_1} a_2 = ka_2$

of xy: $\quad b_3 = \dfrac{b_1}{a_1} a_3 = ka_3$

Hence $b_i = ka_i$ for $i = 1, 2$ and 3.

Hence the coefficients of the highest order terms are proportional.

Let $g(x,y) = (x, kx - y)$ and then

$gof = (a_1 x^2 + \dots + a_4 x + a_5 y, (ka_4 - b_4) x + (ka_5 - b_5) y)$

We can use a "Tschirnhaus" transformation to get rid of the term in xy. Namely let

$$h(x,y) = \left(x - \frac{a_2}{2a_1} y, y\right) \text{ and then}$$

$$gofoh = \left(a_1 x^2 + \frac{4a_1 a_3 - a_2^2}{4a_1} y^2 + a_4 x + \frac{2a_1 a_5 - a_2 a_4}{2a_1} y,\right.$$

$$\left.\frac{b_1 a_4 - a_1 b_4}{a_1} x + \frac{2a_1 (b_1 a_5 - a_1 b_5) - a_2(b_1 a_4 - a_1 b_4)}{2a_1^2} y\right)$$

From the Jacobian, equating coefficients:

$$y: \quad (b_1 a_4 - a_1 b_4)(4a_1 a_3 - a_2^2) = 0$$

$$x: \quad 2a_1 (b_1 a_5 - a_1 b_5) = a_2(b_1 a_4 - a_1 b_4)$$

If $b_1 a_4 - a_1 b_4 = 0$, then we would have $b_1 a_5 = a_1 b_5$ (since $a_1 \neq 0$) and then $b_4 = ka_4$ and $b_5 = ka_5$, which would lead to $|Jf| = 0$ which is not allowed.

Therefore $b_1 a_4 - a_1 b_4 \neq 0$ and thus

$$a_2 = \frac{2a_1\ (b_1\ a_5\ -\ a_1\ b_5\ )}{b_1\ a_4\ -\ a_1\ b_4}$$

ie a_2 is determined by: $(a_1, b_1, a_4, b_4, a_5, b_5)$

Also $4a_1\ a_3 - a_2^2 = 0$ Hence

$$\text{gofoh} = (a_1\ x^2 + a_4\ x + \frac{2a_1\ a_5 - a_2\ a_4}{2a_1}y,\ \frac{b_1\ a_4 - a_1\ b_4}{a_1}\ x),$$

$$\text{then } f = g^{-1}o(a_1\ x^2 + a_4 x + \frac{2a_1\ a_5 - a_2\ a_4}{2a_1}y, \frac{b_1\ a_4 - a_1\ b_4}{a_1}x)oh^{-1}$$

$$f = (x,\ \frac{b_1}{a_1}\ x - y)o(a_1\ x^2 + a_4\ x + \frac{2a_1\ a_5 - a_2\ a_4}{2a_1}y,$$

$$\frac{b_1\ a_4 - a_1\ b_4}{a_1}\ x)o(\ x + \frac{a_2}{2a_1}\ y,y)$$

So f can be written as a composition:

(Linear map) o (Triangular map) o (Linear map)

(where by a Triangular map we mean a map of the form

$$T(x,y) = (x + h(y),y)).$$

Such a triangular map is invertible with inverse

$$(x,y) \longmapsto (x-h(y),y)$$

and so f is an invertible polynomial map and is determined by: $(a_1, b_1, a_4, b_4, a_5, b_5)$. This result proves that "locally" the variety of degree ≤ 2 invertible polynomial maps is 6 dimensional. We shall see later that this is true globally.

From above:

$$f(x,y) = ((a_1^{\frac{1}{2}}\ x + \frac{a_2}{2a_1^{\frac{1}{2}}}\ y)^2 + a_4\ x + a_5\ y,\ k(a_1^{\frac{1}{2}}\ x + \frac{a_2}{2a_1^{\frac{1}{2}}}\ y)^2$$

$$+ b_4 x + b_5 y)$$

So we may write it in the form :

$$f(x,y) =((sx + ty)^2 + a_4\ x + a_5\ y,\ k(sx + ty)^2 + b_4\ x + b_5\ y)$$

provided that $\frac{s}{t} = \frac{b_4 - ka_4}{b_5 - ka_5}$ (since $\frac{2a_1}{a_2} = \frac{b_4 - ka_4}{b_5 - ka_5}$)

and $a_4\ b_5 - a_5\ b_4 \neq 0$

This completes the proof of our theorem.

Remarks

1. We will see later how to allow for the case $t=0$ which strictly is not covered by the above.

2. The complicated calculation to verify the Jacobian Conjecture in this easy case gives an indication of why the Jacobian Conjecture is so hard.

The above procedure paves the way for describing the varieties of invertible polynomial maps. We will carry out similar calculation for the cases of degrees 3 and 4.

The result in the degree ≤ 3 case is:

<u>Theorem</u> A polynomial map of degree ≤ 3 whose Jacobian determinant is a non-zero constant is invertible and is of the form:

$$(x,y) \longmapsto ((s_2 x + t_2 y)^3 + (s_1 x + t_1 y)^2 + a_8 x + a_9 y,$$

$$k(s_2 x + t_2 y)^3 + k(s_1 x + t_1 y)^2 + b_8 x + b_9 y)$$

Where $\dfrac{s_1}{t_1} = \dfrac{s_2}{t_2} = \dfrac{ka_8 - b_8}{ka_9 - b_9}$ and $a_8 b_9 \neq a_9 b_8$

The variety of all such maps is 7 dimensional.

<u>Proof</u> A typical origin-preserving polynomial map of degree ≤ 3 is of the form:

$$f(x,y) = (a_1 x^3 + \ldots\ldots + a_8 x + a_9 y,\ b_1 x^3 + \ldots\ldots + b_8 x + b_9 y)$$

and so the set of all such maps forms a space of dimension 18 and the condition det Jf = constant $\neq 0$ defines 14 equations and one inequality in these coefficients.

As in the degree 2 case above, the leading terms are proportional and we compose on the left with the map:

$$g(x,y) = (x,\ kx - y) \text{ with } k = \frac{b_1}{a_1} \text{ and}$$

compose on the right with the map:

$$h(x,y) = (x - \frac{a_2}{3a_1} y, y)$$

to get a map of the form:

$$g \circ f \circ h = (m_1 x^3 + m_3 xy^2 + m_4 y^3 + m_5 x^2 + m_6 xy + m_7 y^2 + m_8 x + m_9 y,$$

$$n_5 x^2 + n_6 xy + n_7 y^2 + n_8 x + n_9 y)$$

Note that the second component has no degree 3 terms and the first component has no term in x^2y. For future reference we note that:

$a_1 = m_1$; $a_5 = m_5$; $a_8 = m_8$ (coefficients of powers of x) and because the "linear part" of gofoh is g o (linear part of f)o h we must have a matrix equation:

$$\begin{pmatrix} m_8 & m_9 \\ n_8 & n_9 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ k & -1 \end{pmatrix} \begin{pmatrix} a_8 & a_9 \\ b_8 & b_9 \end{pmatrix} \begin{pmatrix} 1 & \frac{-a_2}{3a_1} \\ 0 & 1 \end{pmatrix}$$

leading to $m_9 = -\frac{a_2\, a_8}{3a_1} + a_9$, $n_8 = ka_8 - b_8$ and

$$n_9 = -\frac{a_2}{3a_1}\ (ka_8 - b_8) + (ka_9 - b_9).$$

Then the Jacobian determinant of gofoh is

$$\begin{vmatrix} 3m_1x^2 + m_3y^2 + 2m_5x + m_6y + m_8 & 2m_3xy + 3m_4y^2 + m_6x + 2m_7y + m_9 \\ 2n_5x + n_6y + n_8 & n_6x + 2n_7y + n_9 \end{vmatrix}$$

Since the coefficient of x^3 is zero we get $3m_1\, n_6 = 0$ and since (as in the last theorem) we may assume that $m_1 \neq 0$ we get $n_6 = 0$. Taking this into account and equating the other coefficients in the determinant to zero we get:

x^2y: $\quad 2m_3\, n_5 = 3m_1\, n_7$

xy^2: $\quad m_4\, n_5 = 0$

y^3: $\quad m_3\, n_7 = 0$

$$x^2: \quad 3m_1 n_9 = 2m_6 n_5$$

$$xy: \quad m_3 n_8 + 2m_7 n_5 = 2m_5 n_7$$

$$y^2: \quad m_3 n_9 + 2m_6 n_7 = 3m_4 n_8$$

$$x: \quad 2m_5 n_9 = m_6 n_8 + 2m_9 n_5$$

$$y: \quad m_6 n_9 + 2m_8 n_7 = 2m_7 n_8$$

We now show that $n_5 = 0$
Suppose by way of contradiction that $n_5 \neq 0$. Then $m_4 = 0$ (from the coefficient of xy^2).
From y^3: Either $m_3 = 0$ or $n_7 = 0$
From x^2y: If $m_3 = 0$, then $n_7 = 0$ (since $m_1 \neq 0$)
If $n_7 = 0$, then $m_3 = 0$ (since $n_5 \neq 0$)
and so both $m_3 = 0$ and $n_7 = 0$.
Hence from xy: $m_7 = 0$
From y: $m_6 n_9 = 0$, then either $m_6 = 0$ or $n_9 = 0$.
But $m_6 = 0$ iff $n_9 = 0$ (from x^2)
and so we have $n_9 = 0$ and $m_6 = 0$ anyway.
But then $m_9 = 0$ (from x). But this is a contradiction, since $m_9 n_8 - m_8 n_9 \neq 0$ and so we must have $n_5 = 0$ as claimed.
Since $n_5 = 0$, it follows (from the coefficients of x^2y and x^2) that $n_7 = n_9 = 0$.
From xy: $m_3 = 0$ (since $n_8 \neq 0$, otherwise $|Jf| = 0$ since $n_9 = 0$)

and then $m_3 = m_4 = m_6 = m_7 = 0$. Thus the map gofoh is of the form:

$$(x,y) \longmapsto (m_1 x^3 + m_5 x^2 + m_8 x + m_9 y,\ n_8 x)$$

and so (as in the degree 2 case) f can be written as a composite:

(Linear map) o (Triangular map) o (Linear map)

which is clearly invertible.

Since $n_9 = 0$, it follows from the formula for n_9 that

$$3a_1 (ka_9 - b_9) = a_2 (ka_8 - b_8)$$

We must have $ka_8 - b_8 \neq 0$, otherwise $ka_9 - b_9 = 0$ (since $a_1 \neq 0$) and this would give $|Jf| = 0$.

Therefore $a_2 = \dfrac{3a_1 (ka_9 - b_9)}{ka_8 - b_8}$ and a_2 is determined by: $(a_1, b_1, a_8, b_8, a_9, b_9)$.

Using the formulae for m_i and n_i in terms of a_i and b_i we get

$$f = \left(x, \frac{b_1}{a_1} x - y\right) \circ \left(a_1 x^3 + a_5 x^2 + a_8 x + \frac{3a_1 a_9 - a_2 a_8}{3a_1} y,\right.$$

$$\left.\frac{b_1 a_8}{a_1} - b_8)x\right) \circ \left(x + \frac{a_2}{3a_1} y, y\right).$$

Note that f is determined by:

$(a_1, b_1, a_5, a_8, b_8, a_9, b_9)$

This result indicates that "locally" the variety of degree ≤ 3 invertible polynomial maps is 7 dimensional. We will see later that this is true globally.

From above:

$$f(x,y) = \left(\left(a_1^{\frac{1}{3}} x + \frac{a_2}{3a_1^{\frac{2}{3}}} y\right)^3 + \left(a_5^{\frac{1}{2}} x + \frac{a_2\ a_5^{\frac{1}{2}}}{3a_1} y\right)^2 + a_8x + a_9y,\right.$$

$$\left. k\left(a_1^{\frac{1}{3}} x + \frac{a_2}{3a_1^{\frac{2}{3}}} y\right)^3 + k\left(a_5^{\frac{1}{2}} x + \frac{a_2\ a_5^{\frac{1}{2}}}{3a_1} y\right)^2 + b_8x + b_9y\right)$$

So we may write it in the form:

$$f(x,y) = \left((s_2x + t_2 y)^3 + (s_1x + t_1y)^2 + a_8\ x + a_9\ y,\right.$$

$$\left. k(s_2\ x + t_2y)^3 + k\ (s_1x + t_1y)^2 + b_8x + b_9y\right)$$

for some s_1, t_1, s_2, $t_2 \in \mathbb{C}$ provided that

$$\frac{s_1}{t_1} = \frac{s_2}{t_2} = \frac{ka_8 - b_8}{ka_9 - b_9} \quad \text{(since from the formula for } n_9,$$

$$\left.\frac{3a_1}{a_2} = \frac{ka_8 - b_8}{ka_9 - b_9}\ \right) \quad \text{and } a_8\ b_9 - a_9\ b_9 \neq 0$$

This completes the proof of our theorem.

<u>Remark</u> We will see later how to allow for the case $t_i = 0$ which strictly is not covered by the above.

We now look at the next hardest case: degree ≤ 4. Here there are two possible kinds of invertible polynomial maps and we get the following result:

<u>Theorem</u> Any polynomial map of degree ≤ 4 whose

Jacobian determinant is a non-zero constant is invertible. Any such map is of one of the following types:

1) $(x,y) \longmapsto ((s_3x + t_3y)^4 + (s_2x + t_2y)^3 + (s_1x + t_1y)^2$

$+ a_{13}x + a_{14}y,\ k(s_3x + t_3y)^4 + k(s_2x + t_2y)^3 +$

$k(s_1x + t_1y)^2 + b_{13}x + b_{14}y)$

where $\dfrac{s_1}{t_1} = \dfrac{s_2}{t_2} = \dfrac{s_3}{t_3} = \dfrac{ka_{13} - b_{13}}{ka_{14} - b_{14}}$ and $a_{13} \cdot b_{14} \neq a_{14}\ b_{13}$

2) The composite of a pair of invertible polynomial maps of degree 2.

The maps of type 1) form a variety of dimension 8. Maps of type 2) also form a variety of dimension 8.

Proof A typical origin-preserving polynomial map of degree ≤ 4 is:

$f(x,y) = (a_1\ x^4 + \ldots. + a_{13}x + a_{14}y,\ b_1x^4 + \ldots + b_{13}x + b_{14}y)$

and so the set of all such maps forms a space (over $\mathbb{C}$) of dimension 28 and the condition det Jf = constant $\neq 0$ defines 27 equations and one inequality in these coefficients.

As in the easier two cases, the highest order terms in the Jacobian determinant lead to the conclusion that the highest order terms in the two components are proportional and we may compose on the left with the map

$$g(x,y) = (x,\ kx-y) \text{ where } k = \frac{b_1}{a_1}$$

We compose on the right with the map

$$h(x,y) = (x - \frac{a_2}{4a_1} y, y)$$

to get gofoh (x,y) =

$(m_1x^4 + m_3x^2y^2 + \ldots\ldots + m_{13}x + m_{14}y,\ n_6x^3 + n_7x^2y + \ldots\ldots + n_{13}x + n_{14}y$

As in the last case, $m_2 = 0$ and there are no terms of highest degree in the second component. For future reference we note that:

$$a_1 = m_1;\ a_6 = m_6;\ a_{10} = m_{10};\ n_{10} = \frac{b_1\ a_{10} - a_1\ b_{10}}{a_1}$$

and because the "linear part" of gofoh is g o (linear part of f) o h we have a matrix equation

$$\begin{pmatrix} m_{13} & m_{14} \\ n_{13} & n_{14} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{b_1}{a_1} & -1 \end{pmatrix} \begin{pmatrix} a_{13} & a_{14} \\ b_{13} & b_{14} \end{pmatrix} \begin{pmatrix} 1 & \frac{-a_2}{4a_1} \\ 0 & 1 \end{pmatrix}$$

leading to: $m_{13} = a_{13}$

$$m_{14} = \frac{-a_2\ a_{13}}{4a_1} + a_{14}$$

$$n_{13} = \frac{a_{13}\ b_1}{a_1} - b_{13}$$

and $$n_{14} = -\frac{a_2\ a_{13}\ b_1}{4a_1^2} + \frac{a_2\ b_{13}}{4a_1} + \frac{a_{14}\ b_1}{a_1} - b_{14}$$

The Jacobian determinant of gofoh is

$$\begin{vmatrix} 4m_1\ x^3 + 2m_3\ xy^2 + \ldots\ldots & 2m_3\ x^2\ y + \ldots\ldots\ldots \\ 3n_6\ x^2 + 2n_7\ xy + \ldots\ldots & n_7\ x^2 + \ldots\ldots\ldots\ldots \end{vmatrix}$$

and from the coefficient of x^5 we deduce $n_7 = 0$. Using this fact and calculating all the other coefficients we deduce the following 19 equations:

x^4y: $4m_1 n_8 = 3m_3 n_6$

x^3y^2: $4m_1 n_9 = 3m_4 n_6$

x^2y^3: $m_3 n_8 = 6m_5 n_6$

xy^4: $m_4 n_8 = 6m_3 n_9$

y^5: $3m_4 n_9 = 4m_5 n_8$

x^4: $4m_1 n_{11} = 3m_7 n_6$

x^3y: $4m_1 n_{12} + 3m_6 n_8 = 2m_3 n_{10} + 3m_8 n_6$

x^2y^2: $3m_6 n_9 + m_7 n_8 = 2m_4 n_{10} + 3m_9 n_6$

xy^3: $m_4 n_{11} + 4m_5 n_{10} = 2m_3 n_{12} + 3m_7 n_9$

y^4: $2m_4 n_{12} + 3m_8 n_9 = 4m_5 n_{11} + 3m_9 n_8$

x^3: $4m_1 n_{14} + 3m_6 n_{11} = 2m_7 n_{10} + 3m_{11} n_6$

x^2y: $6m_6 n_{12} + m_7 n_{11} + 4m_{10} n_8 = 2m_3 n_{13} + 4m_8 n_{10} + 6m_{12} n_6$

xy^2: $2m_3 n_{14} + 4m_7 n_{12} + 6m_{10}n_9 + m_{11}n_8 = 3m_4 n_{13} + m_8n_{11}+6m_9n_{10}$

y^3: $m_4n_{14} + 2m_8 n_{12} + 3m_{11} n_9 = 4m_5 n_{13} + 3m_9 n_{11} + 2m_{12}n_8$

x^2: $3m_6 n_{14} + 2m_{10} n_{11} = m_7 n_{13} + 2m_{11} n_{10} + 3m_{14} n_6$

xy: $m_7 n_{14} + 2m_{10} n_{12} + m_{13} n_8 = m_8 n_{13} + 2m_{12} n_{10}$

y^2: $m_8 n_{14} + 2m_{11} n_{12} + 3m_{13} n_9 = 3m_9 n_{13} + 2m_{12} n_{11} + m_{14}n_8$

x: $2m_{10} n_{14} + m_{13} n_{11} = m_{11} n_{13} + 2m_{14} n_{10}$

y: $m_{11} n_{14} + 2m_{13} n_{12} = 2m_{12} n_{13} + m_{14} n_{11}$

We shall show that all the degree 3 terms in the second component vanish. We start with:

<u>Lemma</u> The Coefficient $n_6 = 0$

<u>Proof of Lemma</u>: Assume by way of contradiction that $n_6 \neq 0$. Since $m_1 = a_1 \neq 0$, then from

x^3y^2: $n_9 = \dfrac{3m_4n_6}{4m_1}$

xy^4: $m_4 \left(\dfrac{3m_3 n_6}{4m_1}\right) = 6m_3 \left(\dfrac{3m_4 n_6}{4m_1}\right)$

Hence $m_3 m_4 = 6m_3 m_4$.

Thus $m_3 m_4 = 0$ and either $m_3 = 0$ or $m_4 = 0$

We look at these two cases separately.

<u>Case a)</u> If $m_3 = 0$, then from x^4y: $n_8 = 0$,
from x^2y^3: $m_3 n_8 = 6m_5 n_6$, then $m_5 = 0$
from y^5: either $m_4 = 0$ or $n_9 = 0$
from x^3y^2: $m_4 = 0$ iff $n_9 = 0$ and so both these vanish and we have

$m_3 = m_4 = m_5 = n_8 = n_9 = 0$

<u>Case b)</u> If $m_4 = 0$, then from $x^3 y^2$: $n_9 = 0$,
from y^5: either $m_5 = 0$ or $n_8 = 0$
If $m_5 = 0$, then (from $x^2 y^3$) either $m_3 = 0$ (and we are back to case a)) or $n_8 = 0$. But if $n_8 = 0$ then (from $x^4 y$) $m_3 = 0$ and we are back to case a) anyway.
So in either case we have
m_3 , m_4 , m_5 , n_8 and $n_9 = 0$
and we have exhausted the information of the degree 5 terms.
So we proceed with the lower degree terms.
Now from x^2y^2: $m_9 = 0$
From x^4: $4m_1 n_{11} = 3m_7 n_6$
x^3y: $4m_1 n_{12} = 3m_8 n_6$

and we may cross-multiply and cancel to get

$$m_7 n_{12} = m_8 n_{11}$$

But from xy^2: $4m_7 n_{12} = m_8 n_{11}$ and so $m_7 n_{12} = 0$

Hence either $m_7 = 0$ or $n_{12} = 0$

We treat these two cases separately (Cases c) and d))

Case c) $m_7 = 0$, then we have $n_{11} = 0$ (from x^4).

Then $m_3 = m_4 = m_5 = m_7 = m_9 = n_8 = n_9 = n_{11} = 0$

From x^3y: $4m_1 n_{12} = 3m_8 n_6$

x^3: $4m_1 n_{14} = 3m_{11} n_6$

and we may cross-multiply and cancel to get

$$m_{11} n_{12} = m_8 n_{14}$$

From y^2: $m_8 n_{14} + 2m_{11} n_{12} = 0$, then $3m_8 n_{14} = 0$

and either $m_8 = 0$ or $n_{14} = 0$ (Call these cases C_1) and C_2))

Case C_1) $m_8 = 0$, then $n_{12} = 0$ (from x^3y)

and $m_{12} = 0$ (from x^2y). Then either $m_{11} = 0$ or $n_{14} = 0$ (from y) and from x^3: $m_{11} = 0$ iff $n_{14} = 0$ and so both are zero. Then from x^2: $m_{14} n_6 = 0$ and then $m_{14} = 0$ and so $n_{14} = m_{14} = 0$, which is impossible, since det $Jf = m_{13} n_{14} - m_{14} n_{13} \neq 0$.

So Case C_1) leads to a contradiction.

Case C_2): $n_{14} = 0$

Note that we also have m_3, m_4, m_5, m_7, m_9, n_8, n_9 and $n_{11} = 0$

From x^3: $m_{11} = 0$ and from x^2: $m_{14} = 0$, but this too leads to a zero determinant and so Case C_2) is also impossible.

Now we return to case d).

Case d) $n_{12} = 0$

Note that we also have m_3, m_4, m_5, m_9, n_8, and $n_9 = 0$

From x^3y: $m_8 = 0$ and from y^2: $m_{12}\, n_{11} = 0$

and so either $n_{11} = 0$ or $m_{12} = 0$

If $n_{11} = 0$, from x^4: $m_7 = 0$ and we are reduced to case c). On the other hand, if $m_{12} = 0$, then from x^2y: $m_7\, n_{11} = 0$ and either $m_7 = 0$ or $n_{11} = 0$.

But from x^4: $m_7 = 0$ iff $n_{11} = 0$ and so we are reduced to case c) anyway. Thus $n_6 = 0$. This completes the proof of the lemma.

Corollary of lemma We also have: $n_8 = n_9 = 0$ and

$$\text{gofoh} = (m_1\, x^4 + m_3\, x^2\, y^2 + \ldots\ldots\ldots\ldots + m_{13}\, x + m_{14} y,$$
$$n_{10}\, x^2 + n_{11}\, xy + n_{12}\, y^2 + n_{13}\, x + n_{14}\, y)$$

Proof of corollary: From x^3y: $n_9 = 0$ and from x^4y: $n_8 = 0$

and so as claimed all the degree 3 terms in the second component vanish.

Taking account of this result we may rewrite the equations above:

x^4: $4m_1\, n_{11} = 0$ and so $n_{11} = 0$

x^3y: $2m_1\, n_{12} = m_3\, n_{10}$

x^2y^2: $m_4\, n_{10} = 0$

xy^3: $2m_5 n_{10} = m_3 n_{12}$

y^4: $m_4 n_{12} = 0$

x^3: $2m_1 n_{14} = m_7 n_{10}$

x^2y: $3m_6 n_{12} = m_3 n_{13} + 2m_8 n_{10}$

xy^2: $2m_3 n_{14} + 4m_7 n_{12} = 3m_4 n_{13} + 6m_9 n_{10}$

y^3: $m_4 n_{14} + 2m_8 n_{12} = 4m_5 n_{13}$

x^2: $3m_6 n_{14} = m_7 n_{13} + 2m_{11} n_{10}$

xy: $m_7 n_{14} + 2m_{10} n_{12} = m_8 n_{13} + 2m_{12} n_{10}$

y^2: $m_8 n_{14} + 2m_{11} n_{12} = 3m_9 n_{13}$

x: $2m_{10} n_{14} = m_{11} n_{13} + 2m_{14} n_{10}$

y: $m_{11} n_{14} + 2m_{13} n_{12} = 2m_{12} n_{13}$

From the coefficient of $x^2 y^2$ we get two cases:

$$n_{10} = 0 \text{ and } n_{10} \neq 0$$

These correspond to the two types of map in our theorem.

Case 1

If $n_{10} = 0$, then from x^3 : $n_{14} = 0$ and (since the determinant is a non-zero constant) we must have $n_{13} \neq 0$.

From x^3y: $n_{12} = 0$ (since $m_1 = a_1 \neq 0$)

From x^2y: $m_3\, n_{13} = 0 \implies m_3 = 0$

xy^2: $m_4\, n_{13} = 0 \implies m_4 = 0$

y^3: $m_5\, n_{13} = 0 \implies m_5 = 0$

x^2: $m_7\, n_{13} = 0 \implies m_7 = 0$

xy: $m_8\, n_{13} = 0 \implies m_8 = 0$

y^2: $m_9\, n_{13} = 0 \implies m_9 = 0$

x: $m_{11}\, n_{13} = 0 \implies m_{11} = 0$

y: $m_{12}\, n_{13} = 0 \implies m_{12} = 0$

and so in this case

$$\text{gofoh}\,(x,y) = (m_1\, x^4 + m_6\, x^3 + m_{10}x^2 + m_{13}x + m_{14}y,\ n_{13}x)$$

Hence f can be written as a composite:

(linear map) o (triangular map) o (linear map) which is clearly invertible.

Since $n_{14} = 0$, it follows from the formula for n_{14} that

$$a_2 = \frac{4a_1 \left(\frac{b_1}{a_1} a_{14} - b_{14}\right)}{\frac{b_1}{a_1} a_{13} - b_{13}} \quad \text{and } a_2$$

is determined by $a_1, b_1, a_{13}, b_{13}, a_{14}, b_{14}$.

Using the formulae for m_i and n_i in terms of a_i and b_i we get

$$\text{gofoh} = \left(a_1x^4 + a_6x^3 + a_{10}x^2 + a_{13}x + \frac{4a_1a_{14} - a_2a_{13}}{4a_1} y,\right.$$

$$\left.\frac{b_1\ a_{13} - a_1\ b_{13}}{a_1} x\right)$$

$$f = \left(x, \frac{b_1}{a_1} x - y\right) \circ \left(a_1x^4 + a_6x^3 + a_{10}x^2 + a_{13}x +\right.$$

$$\left.\frac{4a_1\ a_{14} - a_2\ a_{13}}{4a_1} y, \quad \frac{b_1\ a_{13} - a_1\ b_{13}}{a_1} x\right) \circ$$

$$\left(x + \frac{a_2}{4a_1} y,\ y\right)$$

Note that f is determined by:

$a_1, b_1, a_6, a_{10}, a_{13}, b_{13}, a_{14}, b_{14}$.

This result indicates that "locally" the variety of degree ≤ 4 invertible polynomial maps of type 1) is 8 dimensional and we will see later that this is true globally.

From above:

$$f(x,y) = ((a_1^{\frac{1}{4}} x + \frac{a_2}{4a_1^{\frac{3}{4}}} y)^4 + (a_6^{\frac{1}{3}} x + \frac{a_2 \, a_6^{\frac{1}{3}}}{4a_1} y)^3 +$$

$$(a_{10}^{\frac{1}{2}} x + \frac{a_2 \, a_{10}^{\frac{1}{2}}}{4a_1} y)^2 + a_{13}x + a_{14}y,$$

$$k(a_1^{\frac{1}{4}} x + \frac{a_2}{4a_1^{\frac{3}{4}}} y)^4 + k\,(a_6^{\frac{1}{3}} x + \frac{a_2 \, a_6^{\frac{1}{3}}}{4a_1} y)^3 +$$

$$k(a_{10}^{\frac{1}{2}} x + \frac{a_2 \, a_{10}^{\frac{1}{2}}}{4a_1} y)^2 + b_{13}\, x + b_{14}y)$$

and we may rewrite this as

$$f(x,y) = ((s_3\, x + t_3 y)^4 + (s_2 x + t_2 y)^3 + (s_1 x + t_1 y)^2$$

$$+ a_{13}x + a_{14}y,\ k(s_3 x + t_3 y)^4 + k(s_2 x + t_2 y)^3$$

$$+ k\,(s_1 x + t_1 y)^2 + b_{13}\, x + b_{14}\, y)$$

Provided that

$$\frac{s_1}{t_1} = \frac{s_2}{t_2} = \frac{s_3}{t_3} = \frac{ka_{13} - b_{13}}{ka_{14} - b_{14}} \ \left(\text{since } \frac{4a_1}{a_2} = \frac{ka_{13} - b_{13}}{ka_{14} - b_{14}}\right)$$

and $a_{13}\, b_{14} - a_{14}\, b_{13} \neq 0$

This is the form given in the theorem. We will see later how to allow for the case $t_i = 0$ which strictly is not covered by the above.

Case 2 This is the case $n_{10} \neq 0$, $m_4 = 0$. Recall that we also have: n_6, n_7, n_8 and $n_9 = 0$. We first show:

Lemma The coefficient $m_3 = 0$

Proof: Suppose that $m_3 \neq 0$

From the coefficient of $x^3 y$ we get $n_{12} \neq 0$. Then using the coefficients of $x^3 y$ and xy^3 and multiplying we get

$$2m_1\, m_3\, n_{12}^2 = 2m_3\, m_5\, n_{10}^2$$

and hence $m_1\, n_{12}^2 = m_5\, n_{10}^2$ or $\dfrac{m_5}{m_1} = \left(\dfrac{n_{12}}{n_{10}}\right)^2 = s^2$ (say) where $s = \dfrac{n_{12}}{n_{10}} \neq 0$.

From the same two coefficients we also get:

$m_3^2\, n_{10}\, n_{12} = 4m_1\, m_5\, n_{10}\, n_{12}$

and so $m_3^2 = 4m_1\, m_5 = 4m_1^2\, s^2$

So our map gofoh is of the form:

$(x,y) \longmapsto (m_1(x^2 + sy^2)^2 + \text{lower terms},\ n_{10}(x^2+sy^2) + n_{13}x + n_{14}y)$

But composing this map on the left with the invertible map:

$(x,y) \longmapsto (x - \frac{m_1}{n_{10}^2} y^2, y)$ gives a map of the form:

$(x,y) \longmapsto$(polynomial of degree ≤ 3, $n_{10}(x^2+sy^2) + n_{13}x + n_{14}y$)

From the earlier theorem about invertible maps of degree ≤ 3, the first component can have no terms of degree 3 But from the earlier theorem of degree ≤ 2, this map could only be invertible if $s = 0$ which gives us our contradiction.

This completes the proof of the lemma.

Since $m_3 = 0$, we deduce from:

x^3y: $\quad n_{12} = 0$

xy^3: $\quad m_5 = 0$

x^2y: $\quad m_8 = 0$

xy^2: $\quad m_9 = 0$

Hence gofoh $= (m_1x^4 + m_6x^3 + m_7 x^2y + m_{10}x^2 + m_{11}xy + m_{12}y^2$

$$+ m_{13}x + m_{14}y,\ n_{10}x^2 + n_{13}x + n_{14}y)$$

The equations deduced from the Jacobian become:

x^3: $\quad 2m_1 n_{14} = m_7 n_{10}$ 1

xy: $\quad m_7 n_{14} = 2m_{12} n_{10}$ 2

x^2: $$3m_6\, n_{14} = m_7\, n_{13} + 2m_{11}\, n_{10} \qquad \ldots\ldots\ldots\ldots \quad 3$$

x: $$2m_{10}\, n_{14} = m_{11}\, n_{13} + 2m_{14}\, n_{10} \qquad \ldots\ldots\ldots\ldots \quad 4$$

y: $$m_{11}\, n_{14} = 2m_{12}\, n_{13} \qquad \ldots\ldots\ldots\ldots \quad 5$$

From these we deduce:

From 1: $$m_7 = \frac{2m_1\, n_{14}}{n_{10}}$$

2: $$m_{12} = \frac{m_7\, n_{14}}{2n_{10}} = \frac{m_1\, n_{14}^2}{n_{10}^2}$$

Now $n_{14} \neq 0$ otherwise we have $m_7 = m_{12} = 0$ and from 3: $m_{11} = 0$ and then from 4: $m_{14} = 0$ which would give a zero determinant.

From 5: $$m_{11} = \frac{2m_{12}\, n_{13}}{n_{14}} = \frac{2m_1\, n_{13}\, n_{14}}{n_{10}^2}$$

4: $$m_{10} = \frac{m_{11}\, n_{13}}{2n_{14}} + \frac{m_{14}\, n_{10}}{n_{14}} = \frac{m_1\, n_{13}^2}{n_{10}^2} + \frac{m_{14}\, n_{10}}{n_{14}}$$

3: $$m_6 = \frac{m_7\, n_{13}}{3n_{14}} + \frac{2m_{11}\, n_{10}}{3n_{14}} = \frac{2m_1\, n_{13}}{3n_{10}} + \frac{4m_1\, n_{13}}{3n_{10}}$$
$$= \frac{2m_1\, n_{13}}{n_{10}}$$

and so our map gofoh is:

$$(x,y) \longmapsto (m_1\ x^4 + \frac{2m_1\ n_{13}}{n_{10}}\ x^3 + \frac{2m_1\ n_{14}}{n_{10}}\ x^2y +$$

$$\left(\frac{m_1\ n_{13}^2}{n_{10}^2} + \frac{m_{14}\ n_{10}}{n_{14}} \right) x^2 + \frac{2m_1\ n_{13}\ n_{14}}{n_{10}^2}\ xy +$$

$$\frac{m_1\ n_{14}^2}{n_{10}^2}\ y^2 + m_{13}\ x + m_{14}y,\ n_{10}\ x^2 + n_{13}x + n_{14}y)$$

$$= (m_1\ (x^2 + \frac{n_{14}}{n_{10}}\ y)^2 + \frac{2m_1 n_{13}}{n_{10}}\ x^3 + \left(\frac{m_1\ n_{13}^2}{n_{10}^2} + \frac{m_{14}\ n_{10}}{n_{14}}\right)x^2$$

$$+ \frac{2m_1\ n_{13}\ n_{14}}{n_{10}^2}\ xy + m_{13}x + m_{14}y,$$

$$n_{10}\ (x^2 + \frac{n_{14}}{n_{10}}\ y) + n_{13}x).$$

and this is the composition:

$$(x + \frac{m_1}{n_{10}^2}\ y^2,\ y) \circ \left(\frac{m_{14}\ n_{10}}{n_{14}}\ x^2 + m_{13}x + m_{14}y,\ n_{10}x^2 + n_{13}x + n_{14}y\right)$$

From the result on invertible maps of degree ≤ 2 the "right hand" map is invertible. Hence, as the theorem stated: in case 2) the map is the composite of a pair of invertible degree 2 maps. Using the formulae for m_i and n_i in terms of a_i and b_i we get that f is determined by a_1, b_1, a_{10}, b_{10}, a_{13}, b_{13}, a_{14}, b_{14}.

This result indicates that the variety of degree 4 invertible polynomial maps in this second case is 8 dimensional "locally". We shall see later that this is true globally.

This completes the proof of the theorem.

Remark

In the next chapter, we shall see how to avoid doing some of the hard work in the above proofs.

Chapter 3

General Results on Maps with Constant Jacobians

In the calculations in the last chapter it was clear that the highest order terms of a polynomial map with constant Jacobian can be easily dealt with by looking as the highest order terms in the Jacobian determinant.

Theorem (Cf [Vi] pg 415)

Let $f(x,y) = (P_r + P_{r-1} + \ldots\ldots, q_s + q_{s-1} + \ldots\ldots)$ be a non-linear polynomial map with constant Jacobian. If P_r, q_s are the leading terms and are homogeneous polynomials of degrees r, s respectively in x,y then $P_r = c(q_s)^{r/s}$ for some $c \in \mathbb{C}$.

Proof

Let f, P_r and q_s be as in the statement of the theorem. If we look at the terms of highest degree in the determinant of Jf we get: $D_x P_r \cdot D_y q_s - D_x q_s \cdot D_y P_r = 0$. In other words the Jacobian determinant:

$$\frac{\partial(P_r , q_s)}{\partial(x,y)} = 0$$

From the general solution of this partial differential equation we deduce that there is a functional dependence between P_r and q_s. That is, for some (differentiable) function F we have $F(P_r , q_s) = 0$

Since P_r, q_s are polynomials, F is also a polynomial.

We now look at those terms of F which give us terms in x and y of degree k. These must vanish as polynomials in x and y and so we have

$$F_k \ (P_r \ , q_s \ ) = \sum_{ir + js = k} a_{ij} \, P_r^i \ q_s^j = 0$$

Let $P_r^{i_o} \ q_s^{j_o}$ be the term with non-zero coefficient with highest power of P_r. Divide throughout by this and we find:

<u>Lemma</u>: We get a polynomial in t where

$$t = \frac{q_s^{r/d}}{P_r^{s/d}} \qquad \text{and } d = \text{hcf } (r,s)$$

<u>Proof of Lemma</u>: Since $P_r^{i_o} \ q_s^{j_o}$ contains the highest power of P_r it also contains the lowest power of q_s.

Hence after division a typical term is

$$a_{ij} \ \frac{q_s^a}{P_r^b} \quad \text{where } a = j - j_o, b = i_o - i$$

Now since $ri_o + sj_o = ri + sj = k$ we have

$$r(i - i_o) + s \ (j - j_o) = 0 \text{ and so } \frac{r}{s} = \frac{j - j_o}{i_o - i}$$

In its lowest form the left-hand fraction is $\frac{r/d}{s/d}$

and so r/d divides $a = j - j_o$ and s/d divides $b = i_o - i$ and the two quotients are the same. Thus q_s^a / P_r^b is a power of t and the lemma follows.

The polynomial in the lemma factors into linear factors over $\mathbb{C}$ say:

$$e(t-a_1)(t-a_2)\ldots.(t-a_n)$$

and so the polynomial F_k factors into a product of factors of the form

$$q_s^{r/d} - a_i\, p_r^{s/d}$$

As polynomials in x and y one of these must be zero and so we have:

$$p_r = cq_s^{r/s}$$

as in the statement of the theorem.

Remark

In the last chapter when we investigated maps of degree 2, 3, 4 we showed that if the two components had the same degree then the leading terms were proportional. This is just the case r = s of the above theorem.

This last result allows one to attack the Jacobian Conjecture from a different direction, as the following corollary indicates.

Corollary

To prove the Jacobian Conjecture it is enough to prove that for any polynomial map with constant non-zero Jacobian

$$f(x,y) = (p(x,y),\ q(x,y))$$

the degree of q divides the degree of P or vice-versa.

Proof of Corollary

We have (say) the degree of q dividing the degree of P Then in the above theorem $r/s = n$ is an integer and so the leading term of P is a multiple of a power of the leading term of q. So $p-cq^n$ has degree $<$ degree P and we may write

$$f = (x + cy^n , y) \circ f_1$$

with

$$f_1 = (p - cq^n , q)$$

Then degree $f_1 <$ degree f (If degree p = degree q we need to apply this method again to get strict inequality) and f_1 still has a constant non-zero Jacobian. Thus the Jacobian Conjecture will follow by induction on the degree.

This completes the proof of the corollary.

Remark 1)

The proof of the corollary suggests that any invertible polynomial map can be written as a composite of maps of the form $(x,y) \longmapsto (x + cy^n, y)$ and linear maps or equivalently as the composite of linear maps with what, in the last chapter, we called "triangular maps" $(x,y) \longmapsto (x + h(y), y)$ for a polynomial h.

This result is in fact true and is one of the oldest results in this part of the subject. It was proved by Jung in 1942 [J] and has been reproved several times since (eg. Van der Kulk [V] 1953).

Remark 2)

The divisibility condition in the corollary follows in some special cases from a theorem of Magnus [M] 1955: "If the degree of two polynomials are coprime and their Jacobian is constant then this constant is zero"

This proves the Jacobian Conjecture for the case of prime degree. A generalisation of Magnus's theorem by Nakai and Baba [NB] 1977 proves it for degrees which are twice a prime.

Remark 3)

It is not obvious that the divisibility in this corollary holds for any invertible map. In fact Segre [S] (1956) stated the following lemma as part of an attempt to prove the Jacobian Conjecture:

If f, $g \in \mathbb{C}[t]$ are non-constant polynomials and generate the polynomial algebra then degree f divides degree g or vice-versa. This result implies that the divisibility condition holds, since if $(x,y) \longmapsto (p(x,y), q(x,y))$ is invertible with inverse $(x,y) \longmapsto (r(x,y), s(x,y))$ then $x = r(p(x,y), q(x,y))$ and we may replace y by a suitable ux in the identity and use Segre's lemma.

The lemma was proved in 1975 by Abhyankar and Moh [AM] See [BCW] pg 299 for a history of its attempted proof. The paper [AM] also shows that the lemma implies Jung's theorem (See Remark 1)above). We will give a different proof of this fact later in this chapter.

We now show how use of these results can save some of the calculation of the last chapter.

A) The degree is prime (in particular degree = 2 or 3) or the map cannot be written as a composite of lower degree maps.

From the theorem above and Magnus's theorem, either the two components have the same degree - in which case we can compose with a linear map to reduce the degree of one of them, or one of the components (say the second) is linear. In the case $r = 3$ considered in the last chapter this avoids the rather tricky proof that $n_5 = 0$ which enabled us to eliminate the second order terms from the second component.

In the case of a general prime our map is now of the form:

$(x,y) \longmapsto (c_1 (sx + ty)^r + P_{r-1} + \dots, sx + ty)$

and this is the composition:

$(x + c_1 y^r , y) \circ (P_{r-1} + \dots, sx + ty)$

and by the same argument:

$P_{r-1} = c_2 (sx + ty)^{r-1}$

and so on.

Thus our map is:

$(x + h(y) , y) \circ ($ linear map $)$

with h is a polynomial of degree r.

One may then deduce the standard form for invertible polynomial maps of prime degree or for irreducible maps :

$$(x,y) \longmapsto ((s_{r-1}x + t_{r-1}y)^r + \dots + (s_1 x + t_1 y)^2 + ax + by,$$

$$k(s_{r-1} x + t_{r-1}y)^r + \dots + k(s_1 x + t_1 y)^2 + cx + dy)$$

where $\dfrac{s_{r-1}}{t_{r-1}} = \dots = \dfrac{s_1}{t_1} = \dfrac{ka - c}{kb - d}$.

B) The degree 4 case.

In this case we may compose with a linear map to get the second component with degree < 4. If this second component is linear then we are reduced to last case.

From Magnus's theorem we deduce that the degree of the second component cannot be three and this is what we proved "the hard way" in the lemma in the last chapter in which we showed that $n_6 = 0$. The only other possibility is that this second component has degree 2. So we have a map:

$$(x,y) \longmapsto (P_4 + P_3 + P_2 + P_1 , q_2 + q_1)$$

and from the theorem above, $P_4 = cq_2^2$ for some $c \in \mathbb{C}$ and so our map is a composite

$$(x,y) \longmapsto (x + cy^2 , y) \circ (P_3' + P_2' + P_1 , q_2 + q_1)$$

and (as before) since the "right-hand" map is invertible we must have $P_3' = 0$ and then our map is the composite of two degree 2 maps as we showed earlier.

To give an indication of how more complicated cases work we briefly consider:

C) The degree 6 case.

Assume that the first component (say) of our map has degree 6. Then the possibilities are:

1) Our map is of the form

$(x,y) \longmapsto$ (degree 6 polynomial, degree 6 polynomial)

As before, we may compose with a linear map to reduce the degree of the second component and get one of the following cases.

2) We have a map:

$(x,y) \longmapsto$ (degree 6 polynomial, degree 1 polynomial)

This case has been handled in A) above.

3) The map is:

$(x,y) \longmapsto$ (degree 6 polynomial, degree 3 polynomial)

$= (P_6 + \ldots\ldots, q_3 + \ldots\ldots)$

Then $P_6 = cq_3^2$ from our theorem and our map is a composite:

$(x + cy^2 , y) \circ (P_5' + \ldots\ldots, q_3 + \ldots\ldots)$.

From the results about invertible maps of degrees 5 and 4 the right-hand map must in fact be of degree 3 and so our map is a composite of a degree 2 with a degree 3 map.

4) The map is

$(x,y) \longmapsto$ (degree 6 polynomial, degree 2 polynomial)

$= (P_6 + \ldots\ldots, q_2 + \ldots\ldots)$

Then $P_6 = cq_2^3$ and the map is the composite:

$$(x,y) \longmapsto (x + cy^3, y) \circ (P_5' + P_4' + \ldots\ldots, q_2 + q_1)$$

Since the right-hand map is invertible we have $P_5' = 0$ and then $P_4' = c_1 \; q_2^2$ for some $c_1 \in \mathbb{C}$ and so our map is:

$$(x,y) \longmapsto (x + cy^3, y) \circ (x + c_1y^2, y) \circ (P_3'' + \ldots, q_2 + q_1)$$

$$= (x + cy^3 + c_1y^2, y) \circ (P_3'' + \ldots\ldots., q_2 + q_1)$$

and since the right hand map is invertible it follows that $P_3'' = 0$ and our map is a composite of a map of degree 3 with a map of degree 2.

Remark: From Magnus's theorem the degree of the second component cannot be 5. From the generalization of Magnus's theorem [NB] it cannot be 4.

We may use the method outlined in the last case to give a new proof of Jung's theorem from Segre's lemma (different from that in [AM]).

Theorem: Any invertible polynomial map is a composite of linear maps and triangular maps. (Jung [J]).

Proof: We call an "upper triangular map" one of the form

$$(x,y) \longmapsto (x + h(y), y)$$

(The Jacobian matrix of such a transformation is an "upper triangular matrix"). A "lower triangular map" is one of the form $(x,y) \longmapsto (x,y + k(x))$.

Given an invertible map f:

$$(x,y) \longmapsto (P_r + \dots\dots\dots,\ q_s + \dots\dots\dots)$$

then by Segre's lemma (say) s divides r (if not then r divides s and we may compose with the linear map: $(x,y) \longmapsto (y,x)$). Then from the first theorem in this chapter $P_r = cq_s^{r/s}$ and f is the composition:

$$(x + cy^{r/s},\ y) \circ (\acute{P}_{r_1} + \dots\dots,\ q_s + \dots\dots)$$

Either $r_1 < s$ or s divides r_1 and $\acute{P}_{r_1} = c_1\, q_s^{r_1/s}$ and f is the composite:

$$(x + cy^{r/s} + c_1 y^{r_1/s},\ y) \circ (\acute{\acute{P}}_{r_2} + \dots\dots\dots\dots,\ q_s + \dots\dots\dots)$$

Repeating this process gives us a composite of the form:

$$(x + h(y), y) \circ (P_{\acute{r}} + \dots\dots\dots\dots,\ q_s + \dots\dots\dots)$$

with $\acute{r} < s$ and hence $\acute{r}$ dividing s. Use the same method to write f as:

$$(x + h(y), y) \circ (x,\ y + k(x)) \circ (P_{\acute{r}} + \dots\dots\dots\dots,\ q_{\acute{s}} + \dots\dots)$$

with $\acute{s} < \acute{r}$.

We continue this process until we are left with a linear map and our result is complete.

The above proof shows that we have a slightly stronger version of Jung's theorem: Any invertible polynomial map can be written as a product of the form:

$$U_1 \circ L_1 \circ U_2 \circ \dots\dots\dots \circ U_k \circ L_k \circ M$$

with U_i an "upper triangular map", L_i "lower triangular map" and M a linear map. Moreover we may arrange that degree $U_i \geq 2$ for $i \geq 2$ and degree $L_i \geq 2$ for $i < k$.

If we start with a product with a linear map "in the middle" our algorithm produces a new product with the linear map at the end .

eg $\quad (x + y^2 , y) \circ (x, y + x) \circ (x + 2y^2 ,y)$

$$= (x + y^2 , y) \circ (x + 2y^2, y + x + 2y^2)$$

$$= (x + y^2,y) \circ (x + y,y) \circ (-y, x + y + 2y^2)$$

$$= (x + y^2 + y,y) \circ (x,y + 2x^2 - x) \circ (-y,x)$$

We can then show:

Theorem: The degree of our map will be the product of the degrees of the triangular maps.

Proof: Clearly we need only calculate the degree of $U_1 \circ L_1 \circ \ldots\ldots \circ U_k \circ L_k$

We will assume that the "final map" L_k is not the identity (ie the product does not finish with an upper triangular) - otherwise conjugate with the map: $(x,y) \mapsto (y,x)$ which leaves all the degrees unchanged but reverses upper and lower triangular maps.

Lemma: The degree of a product of alternate upper and lower triangular maps starting with a lower triangular map is the degree of a power of x in the second component and the degree of such a product starting with an upper triangular map is the degree of x in the first component provided that no pair of adjacent components have degree one. In each case the degree of the product is the product of the degrees of the components.

Proof of the lemma: This is by induction on the number of maps in the product. So we consider the product U o (p,q) with U the upper triangular map: $(x,y) \longmapsto (x + u(y),y)$ and (p,q) a map whose degree is the degree of a power of x in the second component.

Then this map is $(x,y) \longmapsto (p + u(q),q)$ and the result follows provided that either degree $u \geq 1$ or that p,q have different degrees (otherwise the highest order terms of p and u(q) might cancel one another).

This latter condition will be satisfied provided that no two adjacent components are linear. The other part of the inductive step is to consider products of the form L o (p,q) and the argument is similar.

This completes the proof of the lemma.

The theorem follows from the lemma since the form of our decomposition only allows successive linear maps if the whole product is linear.

Now we look at the composition f o g:

$f = U_1 \circ L_1 \circ \ldots\ldots \circ U_k \circ (L_k\ ) \circ M_1$ where M_1 is a linear map.

$M_1 \circ g = U_{k+1} \circ L_{k+1} \circ \ldots\ldots\ldots \circ U_t \circ (L_t) \circ M_2$ where M_2 is linear map.

So $f \circ g = U_1 \circ L_1 \circ \ldots\ldots \circ U_k \circ (L_k) \circ U_{k+1} \circ \ldots\ldots \circ (L_t) \circ M_2$

1. If L_k is non-linear then no collapse can occur and degree $f \circ g$ = degree f . degree g (Because the lemma in the above theorem still holds if there is a single linear map in the product)

2. If L_k is not present (if L_k is linear include it in M_1) then we have a pair of upper triangular maps coming together and some "collapse" can occur.

a) If degree $U_k \neq$ degree U_{k+1} then the degree of $U_k \circ U_{k+1}$ is the higher of the two degrees and the degree of fog is lower than degree f . degree g by a factor of the lower of these degrees.

Note that degree fog divides degree f . degree g.

b) If degree U_k = degree U_{k+1} the situation is harder. Firstly U_k and U_{k+1} might cancel completely ie $U_k = U_{k+1}^{-1}$ and further collapse of L_{k-1} and L_{k+1} might occur.

Secondly, degree $U_k \circ U_{k+1}$ can be anything from 1 to degree U_K and in this case degree fog will not necessarily divide degree f . degree g

eg. $(x + y^3 + y^2, y) \circ (x - y^3, y) = (x + y^2, y)$

Chapter 4

Standard forms for invertible polynomial maps from $\mathbb{C}^2$ to itself

In this chapter, we develop some standard forms for the invertible polynomial maps of different degrees considered in chapters 2 and 3.

For any polynomial maps f, g we have:
(linear part of f) o (linear part of g) = linear part of fog and hence any invertible polynomial map f with linear part M can be written as $M \circ (M^{-1} \circ f)$: the composite of a linear map with a map whose linear part is the identity. Thus it is enough to write down standard forms for maps with linear part $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Degree ≤ 2

Taking the standard form from chapter 2 and restricting it to the case when the linear part is the identity we get the following standard form for such maps:

$$(x,y) \longmapsto ((sx + ty)^2 + x, - (s/t)(sx + ty)^2 + y)$$

However, this does not allow for the case t = 0 for which we may have a map of the form:

$$(x,y) \longmapsto (x, ex^2 + y) \text{ for some } e \in \mathbb{C}$$

We may combine these two cases in to a single one to get a general form for maps of degree ≤ 2 which preserve the origin and whose linear part is the identity and this general form is:

$$(x,y) \longmapsto (t(sx + ty)^2 + x, - s (sx + ty)^2 + y)$$

for any s, t $\in \mathbb{C}$ (of course the s, t will in general be different from before). Note that any choice of s, t will give such an invertible polynomial map, but that several different choices will give the same map.

In fact, if our map is of the form:

$(x,y) \longmapsto (a_1 x^2 + a_2 xy + a_3 y^2 + x, b_1 x^2 + b_2 xy + b_3 y^2 + y)$

then $t^3 = a_3$ and $- st^2 = b_3$

and so if $a_3 \neq 0$, the number t can be any of the three different cube roots of a_3 and then the number s is determined.

If $a_3 = 0$, then $t = 0$ and from $- s^3 = b_1 \neq 0$, one can take s to be any of the three different cube roots of $-b_1$.

Thus each invertible polynomial map of degree ≤ 2 which preserves the origin and whose linear part is the identity corresponds to exactly three pairs $(s,t) \in \mathbb{C}^2$.

Degree ≤ 3

Again, we consider maps which preserve the origin and whose linear part is the identity. Then from chapter 2, we can determine that such a map is of the form:

$$(x,y) \longmapsto ((s_2 x + t_2 y)^3 + (s_1 x + t_1 y)^2 + x,$$
$$k(s_2 x + t_2 y)^3 + k(s_1 x + t_1 y)^2 + y)$$

with $\quad -\dfrac{s_2}{t_2} = -\dfrac{s_1}{t_1} = k$

However this does not allow for the case where both of the t's are zero for which we may have a map of the form:

$$(x,y) \longmapsto (x,\ e_2\, x^3 + e_1\, x^2 + y)$$

for some e_1, $e_2 \in \mathbb{C}$. We can combine these into a single formula in a way similar to the case (degree $\leq$ 2) by:

$$(x,y) \longmapsto (t_2\,(s_2\, x + t_2\, y)^3 + t_1(s_1\, x + t_1\, y)^2 + x,$$
$$- s_2\,(s_2\, x + t_2\, y)^3 - s_1(s_1\, x + t_1\, y)^2 + y)$$

where now the ratios $s_2:t_2$ and $s_1:t_1$ are the same or undefined. Equivalently $s_2\, t_1 = s_1\, t_2$.

As in the case (degree $\leq$ 2), any suitable values of the numbers s_1, t_1, s_2, t_2, will give such an invertible polynomial map. Again several different choices lead to the same map.

In fact if our map is of the form:

$$(x,y) \longmapsto (a_1\, x^3 + \ldots\ldots + a_4\, y^3 + a_5\, x^2 + \ldots\ldots + a_7\, y^2 + x,$$
$$b_1\, x^3 + \ldots\ldots + b_4\, y^3 + b_5\, x^2 + \ldots\ldots + b_7\, y^2 + y)$$

Then $t_2^4 = a_4$ and $-s_2\, t_2^3 = b_4$

$t_1^3 = a_7$ and $-s_1\, t_1^2 = b_7$

and so if $a_4 \neq 0$ and $a_7 \neq 0$, the number t_2 can be any of the four different fourth roots of a_4 and t_1 can be any of the three different cube roots of a_7 and then the numbers s_1, s_2 are determined.

If $a_4 = 0$, then $a_7 = 0$ also and $t_2 = t_1 = 0$

Then from $-s_2^4 = b_1$, $-s_1^3 = b_5$, we can take s_2 to be any fourth root of $-b_1$ and s_1 any cube root of $-b_5$. Thus each invertible polynomial map of degree ≤ 3 which preserves the origin and whose linear part is the identity corresponds to exactly twelve 4 - tuples:

$(s_1, t_1, s_2, t_2) \in \mathbb{C}^4$

<u>Degree ≤ 4</u>

From chapter 2, we must consider the following two cases:

<u>Case 1</u>: An irreducible invertible polynomial map which preserves the origin and whose linear part is the identity is of the form:

$$(x,y) \longmapsto ((s_3 x + t_3 y)^4 + (s_2 x + t_2 y)^3 + (s_1 x + t_1 y)^2 + x,$$
$$k(s_3 x + t_3 y)^4 + k(s_2 x + t_2 y)^3 + k(s_1 x + t_1 y)^2 + y)$$

with $-s_3/t_3 = -s_2/t_2 = -s_1/t_1 = k$

However this does not allow for the case where the t's are zero for which we may have a map of the form:

$$(x,y) \longrightarrow (x, e_3 x^4 + e_2 x^3 + e_1 x^2 + y)$$

for some $e_1, e_2, e_3 \in \mathbb{C}$. We can combine these into a single formula in a way similar to the other cases by:

$$(x,y) \longmapsto (t_3 (s_3 x + t_3 y)^4 + t_2(s_2 x + t_2 y)^3 + t_1 (s_1 x + t_1 y)^2 + x,$$
$$-s_3(s_3 x + t_3 y)^4 - s_2 (s_2 x + t_2 y)^3 - s_1(s_1 x + t_1 y)^2 + y)$$

where now the ratios $s_3 : t_3$, $s_2 : t_2$ and $s_1 : t_1$ are the same or undefined. Equivalently

$$s_1 t_2 = s_2 t_1$$

$$s_1 t_3 = s_3 t_1$$

and $$s_2 t_3 = s_3 t_2$$

As in the earlier cases, any suitable values of the numbers s_1, t_1, s_2, t_2, s_3, t_3 will give such an invertible polynomial map. Again several different choices lead to the same map.

In fact arguing as above, the pair (s_3, t_3) is determined by the map up to multiplication by a fifth root of unity, the pair (s_2, t_2) up to multiplication by a fourth root and (s_1, t_1) up to multiplication by a cube root.

So each irreducible invertible polynomial map of degree ≤ 4 which preserves the origin and whose linear part is the identity corresponds to exactly sixty 6-tuples:

$$(s_1, t_1, s_2, t_2, s_3, t_3) \in \mathbb{C}^6$$

Case 2

We now consider the other degree 4 maps - those which are products of maps of degree 2.

We shall deal with these as a special case of the following important uniqueness theorem.

Theorem: Any reducible invertible polynomial map f whose linear part is the identity can be written as a product of irreducible maps whose linear parts are all the identity.

If the product of the degrees of the maps in the decomposition is the degree of f then this decomposition is unique.

Proof: We start by writing f as a decomposition of alternate upper and lower triangular maps as in Chapter 3.

If $f = (p,q)$ then there are several cases.

1) degree $p >$ degree q. Then the decomposition starts with a non-linear upper triangular map and we may write $f = U \circ g_1$ where $g_1 = (p_1, q)$ with degree $p_1 <$ degree q.
Then by its construction there is a unique map U with this property. Note, however, that U will not in general have linear part the identity.

2) degree $p <$ degree q. The decomposition starts with a non-linear lower triangular map and we have $f = L \circ g_2$ where $g_2 = (p, q_1)$ with degree $p >$ degree q_1. As before L is uniquely determined by this property.

Case 1) leads to decomposition of the form:

$$U_1 \circ L_1 \circ U_2 \circ \ldots\ldots \circ L_k \circ M$$

with U_i upper triangular, L_i lower triangular and M a linear map.

We can get a decomposition as a product of maps with linear part the identity from this as:

$$(U_1 \circ M_{U_1}^{-1}) \circ (M_{U_1} \circ L_1 \circ M_{L_1}^{-1} \circ M_{U_1}^{-1}) \circ \text{etc}$$

Where M_A is the linear part of the map A. Note that the first factor in this decomposition is a uniquely determined irreducible map with linear part the identity.

Case 2) leads to a similar result and we are left with:

Case 3) $f = (p,q)$ with degree p = degree q.

In this case (see Chapter 3) the decomposition starts with either a (linear) upper traingular map or a (linear) lower triangular map and we must verify that these two different possibilities lead to the same map with linear part the identity. ie suppose that

$$f = M_{L_1} \circ U_1 \circ g_1$$

with M_{L_1} a linear lower triangular map, U_1 an upper triangular map and (as before)

$g_1 = (p_1, q_1)$ with degree $p_1 <$ degree q_1 and also

$f = M_{U_2} \circ L_2 \circ g_2$ with M_{U_2} a linear upper triangular map,

L_2 a lower triangular map and

$g_2 = (p_2, q_2)$ with degree $p_2 >$ degree q_2. From these two we get two maps with linear parts the identity given by:

$A = M_{L_1} \circ U_1 \circ M_{U_1}^{-1} \circ M_{L_1}^{-1}$ and

$B = M_{U_2} \circ L_2 \circ M_{L_2}^{-1} \circ M_{U_2}^{-1}$

and to complete our proof we need to show that these two maps are the same.

So suppose $M_{L_1}(x,y) = (x, kx + y)$

and $U_1 = (h(y) + x, y)$ where $h(y)$ is a polynomial $h(y) = h_n y^n + \dots\dots\dots + h_1 y$.

Then $f = (x, kx + y) \circ (h(y) + x, y) \circ g_1$

$= (h(y) + x, kh(y) + kx + y) \circ g_1$

$= (\frac{1}{k} y + x, y) \circ (-\frac{1}{k} y, kh(y) + kx + y) \circ g_1$

$$= (\tfrac{1}{k}\, y + x, y) \circ (x,\ kh\,(-kx) - kx + y) \circ (\,-\tfrac{1}{k}\, y,\ kx) \circ g_1$$

and so we have $M_{U_2}\ (x,y) = (\frac{1}{k}\, y + x, y)$,

$L_2 = (x,\ kh(\,-kx) - kx + y)$ and $g_2 = (\,-\frac{1}{k}\, y,\ kx) \circ g_1$

and note that the degree of the first component of g_2 is larger than the degree of the second component.

We can now verify that the above maps A, B are the same:

$$A = (x,\ kx + y) \circ (h(y) + x, y) \circ (x - h_1 y,\ y) \circ (x,\ -kx + y)$$

$$= (h(y) + x,\ kh(y) + kx + y) \circ (x + kh_1 x - h_1 y,\ -kx + y)$$

$$= (h(y - kx) + (kh_1 + 1)x - h_1 y,\ kh(y-kx) + k^2 h_1\, x + kx - kh_1 y$$
$$- kx + y)$$

$$= (h(y - kx) + (kh_1 + 1)\, x - h_1 y,\ kh(y - kx) + k^2 h_1\, x + (1 - kh_1) y)$$

$$B = (\tfrac{1}{k}\, y + x, y) \circ (x,\ kh(-kx) - kx + y) \circ (x,\ k^2\, h_1 x + kx + y) \circ$$
$$(-\tfrac{1}{k}\, y + x, y)$$

$$= (h(-kx) - x + \tfrac{1}{k}\, y + x,\ kh\,(-kx) - kx + y) \circ (-\tfrac{1}{k}\, y + x,$$
$$-kh_1 y + k^2 h_1 x - y + kx + y)$$

$$= (h(-kx) + \frac{1}{k} y,\ kh\ (-kx) - Kx + y) \circ (- \frac{1}{k}y + x,\ k^2\ h_1\ x$$

$$+ kx - kh_1 y)$$

$$= (h(y-kx) + kh_1 x + x - h_1 y,\ kh(y - kx) + y - kx + k^2 h_1\ x + kx$$

$$- kh_1 y)$$

$$= (h(y - kx) + (kh_1 + 1)\ x - h_1 y,\ kh(y - kx) + k^2\ h_1\ x$$

$$+ (1 - kh_1)\ y)$$

and so A = B. This completes the proof of the theorem.

Remark

Note that if the product of the degrees of the maps in the decomposition is not equal to the degree of f then the decomposition will not be unique.

eg The product of two upper triangular maps will be another upper triangular one and cancellation may occur as at the end of Chapter 3.

Corollary:

A reducible invertible polynomial map of degree 4 whose linear part is the identity has a unique standard form

$$f_{s_2, t_2} \circ f_{s_1, t_1}$$

where f_{s_i, t_i} is the irreducible polynomial map of degree 2 determined by the pair $(s_i, t_i) \in \mathbb{C}^2$.

Remark:

If we calculate the composite $f_{s_2,t_2} \circ f_{s_1,t_1}$ we get a map with leading terms:

$$(t_2(s_1t_2 - s_2t_1)^2 (s_1x + t_1y)^4 + \ldots\ldots\ldots, - s_2(s_1t_2 - s_2t_1)^2 (s_1x + t_1y)^4 + \ldots\ldots\ldots\ldots)$$

and so the composite will have degree 4 if and only if $s_1t_2 - s_2t_1 \neq 0$.

This completes the consideration of maps of degree ≤ 4.

For maps of higher degree we get a similar situation:

1. The irreducible maps of degree $\leq r$ with linear part the identity have a standard form:

$$(x,y) \longmapsto (t_{r-1}(s_{r-1} x + t_{r-1}y)^r + t_{r-2} (s_{r-2} x + t_{r-2}y)^{r-1} + \ldots. + x, - s_{r-1} (s_{r-1} x + t_{r-1}y)^r - s_{r-2} (s_{r-2} x + t_{r-2}y)^{r-1} - \ldots\ldots + y)$$

where $s_i t_j = s_j t_i$ for each i, j (see Chapter 3).

2. From the last theorem the reducible degree r maps can be written as products of irreducibles in a unique way and thus we get standard forms for such maps.

For example, we have reducible degree 6 maps of the form:

$$f_3 \circ g_2 \quad \text{and } f_2 \circ g_3$$

with f_i and g_i having degree i.

As in the case of the reducible degree 4 maps above we may calculate the leading terms of such compositions and find that if f_i and g_j have standard forms

$$f_i\ (x,y) = (t_{i-1}(s_{i-1}x + t_{i-1}y)^i + \ldots\ldots, - s_{i-1}$$

$$(s_{i-1}\ x + t_{i-1}y)^i - \ldots\ldots.) \qquad \text{and}$$

$$g_j\ (x,y) = (t_{j-1}\ (s_{j-1}x + t_{j-1}\ y)^j + \ldots\ldots., - s_{j-1}$$

$$(s_{j-1}x + t_{j-1}\ y)^j - \ldots\ldots)$$

then the composite will have the correct degree if and only if $s_{i-1}\ t_{j-1} \neq s_{j-1}\ t_{i-1}$.

In compositions of more than two irreducibles the leading terms of the standard forms of each successive pair of irreducibles must satisfy a condition like the above otherwise the degree will be less than the product of the degrees of the components.

Example

Reducible degree 12 maps come in seven different "shapes".

1. degree 6 o degree 2
2. degree 4 o degree 3
3. degree 3 o degree 4
4. degree 2 o degree 6
5. degree 2 o degree 2 o degree 3
6. degree 2 o degree 3 o degree 2
7. degree 3 o degree 2 o degree 2

Remark:

Reducible maps of degree r come in K different shapes where K is the number of ways of writing r as a product of proper factors - taking order into account. These seems to be no easy formula for K.

Chapter 5

Spaces and Varieties for invertible polynomial maps

Using the standard forms in the last chapter, we will describe the varieties formed by invertible polynomial maps.

1. Maps of degree 1

Since only maps which preserve the origin have been considered, these form the space $GL(2, \mathbb{C})$ which can be regarded as an open subset of $\mathbb{C}^4$. It is therefore an affine variety of dimension 4.

2. Maps of degree ≤ 2

From Chapter 4 a standard form for such a map which has linear part the identity is

$$(x,y) \longmapsto (t(sx + ty)^2 + x, \; - s(sx + ty)^2 + y)$$

and such maps form a variety that we will call V_2. From the calculations of Chapter 4 such a map determines the pair $(s,t) \in \mathbb{C}^2$ up to multiplication by a cube root of unity.

Thus $V_2 \cong \mathbb{C}^2/\sim$ where $\sim$ is the action of the cyclic group C_3 of order 3 where a cube root w of unity acts by

$$(s,t) \longmapsto (ws, wt)$$

We will now show how to get the space of all invertible polynomial maps of degree ≤ 2 from those whose linear part is the identity.

Given any invertible polynomial map of degree ≤ 2 it will have a linear part $M \in GL(2, \mathbb{C})$ and the map $M^{-1} \circ f$ will have linear part the identity.

Thus such a map determines a point:

$$(M, M^{-1} \circ f) \in GL(2,\mathbb{C}) \times V_2$$

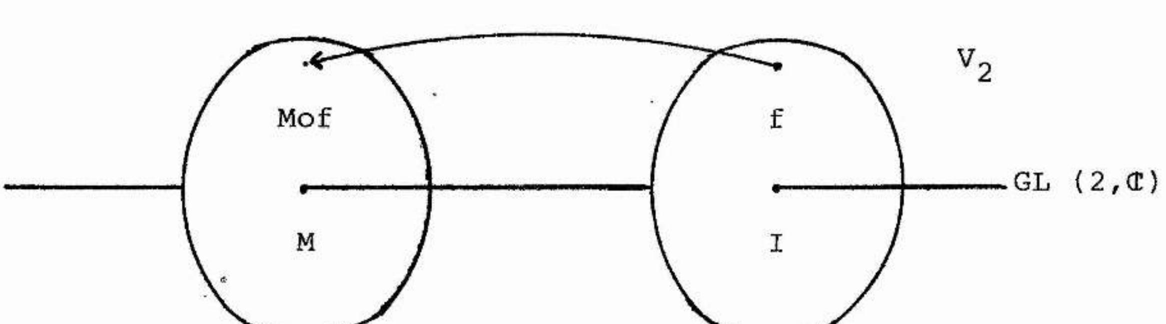
Moreover any point $(M, g) \in GL(2, \mathbb{C}) \times V_2$ determines a map $M \circ g$ of degree ≤ 2. Since these two correspondences are mutually inverse the following has been proved:

Theorem: The space of all invertible polynomial maps of degree ≤ 2 is

$$GL(2, \mathbb{C}) \times \mathbb{C}^2/C_3$$

Remark: This is a space of dimension 6 and the subspace of linear maps is included as the "spine" $GL(2, \mathbb{C}) \times (o,o)$

This space may be represented schematically as:



The action of $GL(2,\mathbb{C})$ takes us from one "fibre" to another.

3. Maps of degree ≤ 3

Here a standard form for such a map with linear part the identity is:

$$(x,y) \longmapsto (t_2\ (s_2x + t_2y)^3 + t_1\ (s_1x + t_1y)^2 + x,$$

$$- s_2\ (s_2x + t_2y)^3 - s_1\ (s_1x + t_1y)^2 + y)$$

where $s_1\ t_2 = s_2\ t_1$ and now the pairs

$(s_2\ ,\ t_2)$ and $(s_1\ ,\ t_1\ ) \in \mathbb{C}^2$

are determined up to multiplication by fourth and cube roots of unity respectively.

So the variety V_3 of all such maps is $U/\sim$ where U is the affine variety $V(x_1\ x_4 - x_2\ x_3) \subset \mathbb{C}^4$ and $\sim$ is the action of the group $C_3 \times C_4$ on $\mathbb{C}^4$ and hence on U given by

$$(s_1\ ,\ t_1\ ,\ s_2\ ,\ t_2\ ) \longmapsto (w_1s_1\ ,\ w_1t_1\ ,\ w_2s_2\ ,\ w_2\ t_2)$$

where $(w_1\ ,\ w_2\ ) \in C_3 \times C_4$ is a pair of roots of unity.

Then as above we can prove:

Theorem: The space of all invertible polynomial maps of degree ≤ 3 is

$$GL\ (2,\ \mathbb{C}) \times V_3$$

Remark: This is a space of dimension 7 since V_3 is 3 - dimensional. Note that V_3 contains V_2 as the subspace of points of the form:

$$(s_1 , t_1 , o , o)$$

4. Maps of degree ≤ 4

Here things are more complicated since there are two possible forms for such maps.

First, consider the "irreducible" maps of degree 4 which cannot be written as the composition of two degree 2 invertible polynomial maps.

The standard form for such a map with linear part the identity is:

$$(x,y) \longmapsto (t_3(s_3x + t_3y)^4 + t_2(s_2x + t_2y)^3 + t_1(s_1x + t_1y)^2 + x,$$

$$- s_3(s_3x + t_3y)^4 - s_2(s_2x + t_2y)^3 - s_1(s_1x + t_1y)^2 + y)$$

where the racios $s_3:t_3$, $s_2:t_2$ and $s_1:t_1$ are equal.

i.e. $s_1 t_2 = s_2 t_1$, $s_1 t_3 = s_3 t_1$ and $s_2 t_3 = s_3 t_2$

We need all these three equations because if a pair (s_i , t_i) is (o,o) then two of the equations will be satisfied and we will need the third to specify the ratio between the others.

So the variety, V_4 of all such maps is $W/\sim$ where W is the affine variety

$$V\ (x_1 x_4 - x_2 x_3\ ,\ x_1 x_6 - x_2 x_5\ ,\ x_3 x_6 - x_4 x_5\ ) \subset \mathbb{C}^6$$

and $\sim$ is the action of the group $C_3 \times C_4 \times C_5$ on $\mathbb{C}^6$ and hence on W given by

$$(s_1, t_1, s_2, t_2, s_3, t_3) \longmapsto (w_1 s_1, w_1 t_1, w_2 s_2, w_2 t_2, w_3 s_3, w_3 t_3)$$

where $(w_1, w_2, w_3) \in C_3 \times C_4 \times C_5$ as above.

Note that the variety W has dimension 4 since at points where $x_i \neq 0$ the three equations defining W can be reduced to two independent ones. Then as above we can prove

<u>Theorem</u>: The space of all irreducible invertible polynomial maps of degree ≤ 4 is

$$GL\ (2,\ \mathbb{C}) \times V_4$$

<u>Remark 1</u>: This is a space of dimension 8 since V_4 is 4 - dimensional. Note that V_4 contains V_3 as the subspace of points of the form $(s_1\ ,\ t_1\ ,\ s_2\ ,\ t_2\ ,\ o\ ,\ o\ )$.

<u>Remark 2</u>: Similarly (from Chapter 4) for invertible polynomial maps of degree $\leq$ r with linear parts the identity

which are irreducible i.e. which cannot be written as a composition of maps of lower degree, one may show the space of all such maps is of the form GL $(2, \mathbb{C}) \times V_r$ where V_r is defined in a way similar to V_2, V_3 and V_4 above.

V_r will have dimension r and so the space of all irreducible invertible polynomial maps will have dimension r + 4.

As above we get inclusions:

$$V_2 \subset V_3 \subset \ldots\ldots \subset V_r \subset \ldots\ldots\ldots$$

Now we look at maps which are not irreducible. First we consider the maps of degree 4 which are products of two degree 2 invertible polynomial maps with linear parts the identity.

The standard form for such a map is

$$(x,y) \mapsto (t_2\ ((s_2t_1 - s_1t_2)\ (s_1x + t_1y)^2 + (s_2x + t_2y))^2 +$$

$$t_1(s_1x + t_1y)^2 + x,\ - s_2((s_2t_1 - s_1t_2)\ (s_1x + t_1y)^2 +$$

$$(s_2x + t_2y))^2 - s_1(s_1x + t_1y)^2 + y)$$

where $s_2t_1 - s_1t_2 \neq 0$ otherwise the map would have degree < 4. Such maps form a variety that we will call V_4'.

Recall that in chapter 4 we proved that this decomposition into two degree 2 maps is unique and so we can identify V_4' with a subset of $V_2 \times V_2$ by mapping $g \in V_4'$ to the uniquely determined $(f_{s_2,\, t_2},\, f_{s_1,\, t_1}) \in V_2 \times V_2$.

Given a pair $(f_{s_2,t_2},\, f_{s_1,t_1}) \in V_2 \times V_2$ for which $s_1\, t_2 \neq s_2\, t_1$ we can map it to the composite in V_4'.

So V_4' can be identified with the space $V_2 \times V_2 - D$ where D is the subset of $V_2 \times V_2$ consisting of pairs $(f_{s_1,\, t_1},\, f_{s_2,\, t_2})$ with $s_1\, t_2 = s_2\, t_1$.

Note that the variety U_4 of all invertible polynomial maps of degree ≤ 4 with linear part the identity is the union of V_4 and V_4'. To describe its structure more exactly we must say have these two components are attached.

From the formula above note that if

$$s_1\, t_2 = s_2\, t_1$$

then the composite of $f_{s_1,\, t_1}$ and $f_{s_2,\, t_2}$ is an element of degree ≤ 2 which therefore lies in the subset V_2 of V_4 consisting of such maps. So we define an attaching map h from D onto the subset V_2 by

$$h(f_{s_1,\, t_1},\, f_{s_2,\, t_2}) = f_{s_2,\, t_2} \circ f_{s_1,\, t_1}$$

Then $U_4 = V_4 \cup_h (V_2 \times V_2)$.

Note that the subset D of $V_2 \times V_2$ is a three-dimensional set while the subset V_2 of V_4 to which it is attached is only two-dimensional. So each point of V_2 will be attached to a one-dimensional subset of $V_2 \times V_2$. The structure of this one-dimensional subset depends on which point of V_2 we consider.

Then as before we can handle the maps with any linear part to get the result

<u>Theorem</u>: The space of all invertible polynomial maps of degree ≤ 4 can be identified with

$$GL(2, \mathbb{C}) \times U_4$$

where U_4 is the space $V_4 \cup_h (V_2 \times V_2)$ described above.

<u>Remark</u>: This space has dimension 8 since the space U_4 has dimension 4.

Now we consider the maps of degree 6 which are either products of degree 2 and degree 3 irreducible maps, or degree 3 and degree 2 with linear parts the identity. From the uniqueness theorem in Chapter 4 each of these cases has a unique standard form as follows:

1. $f_{s_1,t_1} \circ f_{s_2, t_2, a_1, b_1}$

$$= (t_1(s_1x + t_1y)^2 + x, - s_1(s_1x + t_1y)^2 + y) \circ (t_2(s_2x + t_2y)^3 +$$

$$b_1(a_1x + b_1y)^2 + x, - s_2(s_2x + t_2y)^3 - a_1 (a_1x + b_1y)^2 + y)$$

$$= (t_1(s_2t_1 - s_1t_2)^2 (s_2x + t_2y)^6 + \dots\dots\dots + x,$$

$$-s_1(s_2t_1 - s_1t_2)^2 (s_2x + t_2y)^6 - \dots\dots\dots + y)$$

2. $f_{s_2, t_2, a_1, b_1} \circ f_{s_1, t_1}$

$$= (t_2(s_2t_1 - s_1t_2)^3 (s_1x + t_1y)^6 + \dots\dots\dots\dots + x,$$

$$- s_2(s_2t_1 - s_1t_2)^3 (s_1x + t_1y)^6 - \dots\dots\dots\dots + y)$$

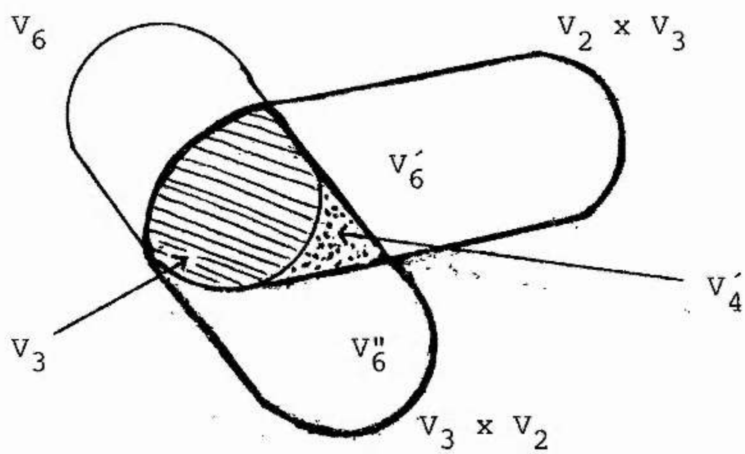
where $a_1 t_2 = b_1 s_2$ (see Chapter 4) and $s_2 t_1 - s_1 t_2 \neq 0$ otherwise the map would have degree less than 6. Such maps form two varieties that we will call V_6', V_6'' respectively.

Since these decompositions are unique we can identify V_6' with a ~~subset~~ subset of $V_2 \times V_3$ and V_6'' with a subset of $V_3 \times V_2$ as in degree 4 case.

In the case $s_2t_1 = s_1t_2$ where the composite has degree less than 6 we have two possibilities. Either the map is irreducible (in which case by Chapter 3 it has degree ≤ 3) or it is a reducible map (and must have degree 4 and be the composite of two degree 2 maps).

Hence the variety U_6 of all invertible maps of degree ≤ 6 (with linear part the identity) is formed from the union of V_6 (the irreducible maps of degree ≤ 6), $V_2 \times V_3$ and $V_3 \times V_2$ by identifying the maps in $V_2 \times V_3$ and $V_3 \times V_2$ with $s_1 t_2 = s_2 t_1$ with the appropriate map in $V_3 \subset V_6$ (if the maps are irreducible) and by identifying the reducible maps of degree 4 in $V_2 \times V_3$ and $V_3 \times V_2$ with one another.

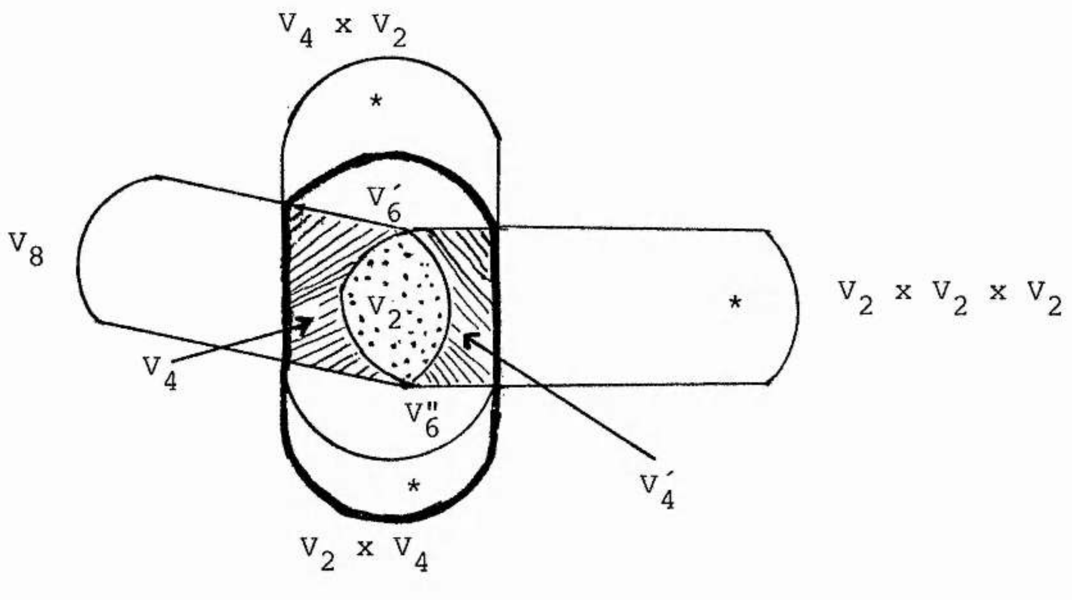
So U_6 is the disjoint union of V_6, V_6', V_6'' and a copy of the variety V_4' of reducible degree 4 maps. These are attached as the following schematic diagram indicates:



For higher degree the situation gets more complicated. For maps of degree ≤ 7 (since maps of degree 7 are all irreducible) the variety of all such maps is the union of V_7 (irreducible maps of degree ≤ 7), V_6', V_6'' and V_4' attached in a way similar to the above.

For maps of degree 8 the reducible maps come in three different kinds: composites of degree 2 with degree 4 irreducibles, composites of degree 4 with degree 2 and composites of three degree 2 maps.

The variety of degree $\leq$ 8 maps is the union of V_8 (irreducible degree $\leq$ 8 maps) with these three and with V_6', V_6'' and V_4'. It can be formed from V_8, $V_4 \times V_2$, $V_2 \times V_4$ and $V_2 \times V_2 \times V_2$ by making appropriate identifications as the following diagram indicates.



* = 3 different kinds of reducilbe degree 8 maps.

Then as before we can handle the maps with any linear part to get the general result

<u>Theorem</u>: The space of all invertible polynomial maps of degree $\leq$ r can be identified with

$$GL(2, \mathbb{C}) \times U_r$$

where U_r is a variety of all invertible polynomial maps of degree $\leq r$ with linear part the identity.

Remark: This space has dimension $r + 4$ since the space U_r has dimension r.

Some group actions on the varieties of invertible maps

We now note some interesting group actions on the varieties we have been considering.

1) For all the varieties U_r (the varieties of invertible maps of degree $\leq r$ with linear part the identity) the map $f \longmapsto f^{-1}$ gives an action of the cyclic group C_2.

On the variety $V_r \subset U_r$ of irreducible maps of degree $\leq r$, from the standard form we get:

$$(t_{r-1}(s_{r-1}x + t_{r-1}y)^r + \dots + x, -s_{r-1}(s_{r-1}x + t_{r-1}y)^r - \dots + y)^{-1}$$

$$= (-t_{r-1}(s_{r-1}x + t_{r-1}y)^r - \dots + x, s_{r-1}(s_{r-1}x + t_{r-1}y)^r + \dots + y)$$

and so the action on V_r arises from an action of C_2 on the space

$$\mathbb{C}^{2r-2}/C_{r+1} \times C_r \times \dots \times C_3$$

introduced earlier.

The action of the non trivial element of C_2 on the element

$$((s_{r-1}, t_{r-1}),(s_{r-2}, t_{r-2}),\ldots,(s_1,t_1)) \in \mathbb{C}^2/_{C_{r+1}} \times$$

$$\mathbb{C}^2/_{C_r} \times \ldots \times \mathbb{C}^2/_{C_3}$$

maps it to

$$((w_{r+1}\, s_{r-1}, w_{r+1}\, t_{r-1}),\ldots\ldots, (w_3\, s_1\, , w_3\, t_1))$$

where w_k is a k-th root of -1.
ie C_2 acts on each factor $\mathbb{C}^2/_{C_k}$ by the action of the generator of C_{2k} in the usual action of C_{2k} on $\mathbb{C}^2$.

For the variety V_2 this is just multiplication by -1 and the action arises from an action of C_2 on $\mathbb{C}^2$ but this does not happen in general.

Note, however, that on the variety of reducible maps in general this action will change the order of factors. For example, in the variety U_8 just considered, the inverse of a map which is the composite of a degree 4 map with a degree 2 map is the composite of a degree 2 map with a degree 4 map.

2) The other group with an interesting action on our varieties is the group $GL(2,\mathbb{C})$ acting via conjugation.

Since the conjugate of a map with linear part the identity also has linear part the identity this does give an action on U_r.

We start with the action on the varieties V_r of irreducibles.

As above we can regard V_r as a subset of

$$\mathbb{C}^{2r-2}/C_{r+1} \times \ldots \times C_3 = \mathbb{C}^2/C_{r+1} \times \ldots\ldots\ldots \times \mathbb{C}^2/C_3$$

We get:

Theorem: The action of an element A of $GL(2,\mathbb{C})$ on V_r is induced by the actions of A on each factor $\mathbb{C}^2/C_k$ given by:

$$(s,t) \longmapsto \Delta_A^{-\frac{1}{k}} \quad M_A^{tr}\ (s,t)$$

Where M_A is the matrix of A and Δ_A is the determinant of A.

Proof:

Note that the ambiguity of the k-th root does not matter for the quotient space.

Take $A(x,y) = (ax + by, cx + dy)$ with

$$A^{-1}(x,y) = \frac{1}{\Delta_A} (dx - by, - cx + ay)$$

Let $f \in V_r$ be of the form:

$$(x,y) \longmapsto (\ldots\ldots + t(sx + ty)^{k-1} + \ldots\ldots\ldots + x, \ \ldots\ldots -s(sx + ty)^{k-1}$$

$$- \ldots\ldots + y)$$

where the pair $(s,t) \in \mathbb{C}^2/_{C_k}$ determines the terms of degree k-1.

Since A and A^{-1} are linear the terms of degree k-1 in the conjugate are formed by conjugating these.

So calculating:

$$A^{-1} \circ (\ldots\ldots + t\ (sx + ty)^{k-1} + \ldots\ldots, \ldots\ldots - s(sx + ty)^{k-1}$$

$$- \ldots\ldots) \circ A$$

gives:

$$\frac{1}{\Delta_A}(dx - by, - cx + ay) \circ (\ldots\ldots + t\ (asx + bsy + ctx + dty)^{k-1}$$

$$+ \ldots\ldots, \ \ldots\ldots - s\ (asx + bsy + ctx + dty)^{k-1} + \ldots\ldots)$$

$$= \frac{1}{\Delta_A} (\ldots\ldots + (bs + dt)((as + ct)x + (bs + dt)y)^{k-1} + \ldots\ldots,$$

$$\ldots\ldots - (as + ct)((as + ct)x + (bs + dt)\ y)^{k-1} + \ldots\ldots)$$

and this determines the point $(s_1, t_1) \in \mathbb{C}^2/_{C_k}$

given by:

$$s_1^k = \frac{(as + ct)^k}{\Delta_A} \quad , \quad t_1^k = \frac{(bs + dt)^k}{\Delta_A}$$

$$\text{ie} \quad s_1 = \Delta_A^{-\frac{1}{k}} \; (as + ct) \; , \qquad t_1 = \Delta_A^{-\frac{1}{k}} \; (bs + dt)$$

which is the result in the theorem.

The action on the rest of the elements in U_r is given by:

<u>Theorem</u>: The result of the action of GL $(2, \mathbb{C})$ on an element which is the product of irreducibles is given by the product of the results of the actions of GL $(2, \mathbb{C})$ on the irreducible factors.

<u>Proof</u>: A acts on $g = f_1 \circ f_2 \circ \ldots\ldots \circ f_n$ by

$$g \longmapsto A^{-1} \circ g \circ A = A^{-1} \circ f_1 \circ A \circ A^{-1} \circ f_2 \circ \ldots\ldots \circ A \circ A^{-1} \circ f_n \circ A$$

which is the composite of what we get when A acts on the irreducible factors.

CHAPTER 6

Elements of finite order in the space of invertible polynomial maps

We now consider the problem of identifying the elements of finite order in the space of invertible polynomial maps of a given degree.

Since conjugation by any invertible element preserves order, the most obvious kind of map with finite order is a conjugate of a linear map of finite order - of which there are many in GL $(2, \mathbb{C})$. (From the usual eigen value description these are all conjugate to $\begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}$ where w_1 and w_2 are n-th roots of unity.)

One main result is:

Theorem 1

All elements of finite order are conjugate to linear maps of finite order.

We first consider the case of an irreducible map.

Theorem 2

Let f be a polynomial map of degree r which is not a composition of maps of lower degree. If f has finite order n, then f is conjugate to a linear map.

Proof: We can write f as:

Mog with M linear and g having linear part the identity. The linear map M has finite order n also and so is conjugate to a map of the form $\begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}$ with w_1, w_2 n- th roots of unity. Hence f is conjugate to $f_1 = \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix} \circ g_1$ (where g_1 still has linear part the identity). Take the standard form for g_1 (see Chapter 4) and we get:

$$f_1(x,y) = (w_1 t_1 (s_1 x + t_1 y)^k + \ldots\ldots + w_1 x, - w_2 s_1 (s_1 x + t_1 y)^k + \ldots\ldots + w_2 y)$$

Assume for the moment that $s_1 \neq 0$ (we will handle the case $s_1 = 0$ later).

Now conjugate f_1 by the linear map:

$N(x,y) = (x + \frac{t_1}{s_1} y, y)$ with inverse

$N^{-1}(x,y) = (x - \frac{t_1}{s_1} y, y)$ to get:

$$f_2(x,y) = (t_1 (w_1 - w_2) s_1^k x^k + \frac{w_1 s_1 t_2 - w_2 t_1 s_2}{s_1} s_2^{k-1} x^{k-1} + \ldots\ldots\ldots + w_1 x + \frac{t_1}{s_1}(w_2 - w_1)y, - w_2 s_1^{k+1} x^k - w_2 s_2^k x^{k-1} + \ldots\ldots + w_2 y)$$

Since f_2 is a conjugate of f_1 it too has finite order n. Hence we can apply:

<u>Lemma</u> 1 A polynomial map f of the form:

$$(x,y) \longmapsto (ax^k + \text{lower terms}, \ldots\ldots..)$$

with $a \neq 0$ cannot have finite order.

Proof of Lemma 1 The map fofo .... of has the leading term of its first component a power of x which will not vanish if $a \neq 0$. Hence it could not be the identity map. The lemma follows.

Thus we have $t_1 = 0$ or $w_1 = w_2$ since $s_1 \neq 0$

In the case $t_1 = 0$:

f_2 reduces to the map $(x,y) \longmapsto (w_1x,\ q(x) + w_2y)$

for some polynomial q.

In the case $w_1 = w_2$:

f_2 reduces to the same form (since $s_1\ t_2 = t_1\ s_2$, $s_1\ t_3 = s_3\ t_1$, etc.(see Chapter 4).

If $s_1 = 0$ (This was the case not treated earlier):

f_1 is of the form $(x,y) \longmapsto (p(y) + w_1\ x,\ w_2\ y)$

The other two cases can be reduced to this one by conjugating by the map $(x,y) \longmapsto (y,x)$ and so it is enough to treat this case only as follows:

Lemma 2

Let f be a map of the form

$$f\ (x,y) = (p(y) + w_1x,\ w_2y)$$

where w_1 and w_2 are n-th roots of unity. Then f has finite order if and only if the coefficient of y^r in p(y) is zero for any r for which $w_2^r = w_1$.

Proof of lemma 2 : Forming the composition fofo.......of (n times) gives the map:

$$(x,y) \longmapsto (p(w_2^{n-1}\ y) + w_1\ p\ (w_2^{n-2}\ y) + \ldots . + w_1^{n-1}\ p(y) +$$

$$w_1^n\ x,\ w_2^n\ y)$$

and so this is the identity iff

$$p(w_2^{n-1}\ y) + w_1\ p(w_2^{n-2}\ y) + \ldots\ldots\ldots + w_1^{n-1}\ p(y) = 0$$

If $p(y) = a_k\ y^k + a_{k-1}\ y^{k-1} + \ldots\ldots\ldots + a_2y^2$ (we can assume there are no linear or constant terms) this condition becomes

$$a_k\ y^k\ (w_2^{(n-1)k} + w_1\ w_2^{(n-2)k} + \ldots\ldots . + w_1^{n-1}) +$$

$$a_{k-1}\ y^{k-1}(w_2^{(n-1)(k-1)} + w_1\ w_2^{(n-2)(k-1)} + \ldots . . +$$

$$w_1^{n-1}) + \ldots\ldots = 0$$

Each of these coefficients must vanish and so putting

$u = w_2^{k-i}$, $v = \frac{1}{w_1}$ we get equations

$a_{k-i}\ w_1^{n-1}\ (u^{n-1}\ v^{n-1} + u^{n-2}\ v^{n-2} + \ldots\ldots +1) = 0$ for

$i = 0, 1, \ldots\ldots, k-2$.

Now uv is also an n-th root of unity and so the term in brackets vanishes provided that $uv \neq 1$. If $uv = 1$ then the corresponding a_i must vanish as in the statement of the lemma.

We now show:

Lemma 3:

If $f(x,y) = (p(y) + w_1 x, w_2 y)$ has finite order then f is conjugate to the linear map $(x,y) \longmapsto (w_1 x, w_2 y)$.

Proof of Lemma 3: Consider the conjugate of the linear map $(x,y) \longmapsto (w_1 x, w_2 y)$ by the triangular map

$(x,y) \longmapsto (q(y) + x, y)$ with inverse

$(x,y) \longmapsto (-q(y) + x, y)$

This gives us a map:

$(x,y) \longmapsto (q(w_2 y) - w_1 q(y) + w_1 x, w_2 y)$

and we will show that we can choose q so that $q(w_2 y) - w_1 q(y) = p(y)$.

So suppose $p(y) = a_k y^k + a_{k-1} y^{k-1} + \ldots\ldots + a_2 y^2$ and

take $q(y) = b_k y^k + b_{k-1} y^{k-1} + \ldots\ldots + b_2 y^2$. Then we must have: $b_i (w_2^i - w_1) = a_i$ for each i.

From Lemma 2 a_i is only non-zero if $w_2^i - w_1 \neq 0$ and so we can solve all these equations.

This completes the proof of lemma 3. Hence our theorem is proved.

Remark: The proof also shows that an irreducible map of degree r which happens to have finite order is formed by conjugation of a linear map of finite order by an irreducible map of degree r.

Corollary: All maps of the forms

$$(x,y) \longmapsto (p(y) + wx,\ y)$$

and

$$(x,y) \longmapsto (x,\ q(x) + wy)$$

have finite order n where w is a non-trivial n-th root of unity for $n \geq 2$.

Proof: The first case follows from Lemma 2 above. The second map is conjugate to the first by the map $(x,y) \longmapsto (y,x)$.

As an illustration of the way Lemmas 2 and 3 above work, we consider a particular example in some detail.

Example: Consider a map

$$f(x,y) = (p(y) + w_1\, x,\ w_2 y)$$

of finite order 4 where

$$w_1 = -1 \quad , \quad w_2 = i$$

and $p(y) = ay^4 + by^3 + cy^2$.

The condition that f has finite order 4 is:

$$p(w_2^3\, y) + w_1\, p\,(w_2^2\, y) + w_1^2\, p\,(w_2 y) + w_1^3\, p(y) = 0$$

and so

$$p\,(-iy) - p\,(-y) + p\,(iy) - p(y) = 0$$

Hence

$$ay^4 + ib\,y^3 - cy^2 - ay^4 + by^3 - cy^2 + ay^4 - iby^3 - cy^2 -$$

$$ay^4 - by^3 - cy^2 = - 4\,cy^2 .$$

So the polynomial p satisfies our condition if and only if $c = 0$ and hence
$f(x,y) = (ay^4 + by^3 - x,\ iy)$ has order 4. Now we may assume that

$$f(x,y) = (q(y) + x, y) \circ (-x,\ iy) \circ (-q\,(y) + x,\ y)$$

$$= (q(iy) + q\,(y) - x,\ iy)$$

where $q\,(y) = my^4 + dy^3$.

Thus $q\,(iy) + q\,(y) = ay^4 + by^3$ and so

$$m\,(iy)^4 + d\,(iy)^3 = ay^4 + by^3 - my^4 - dy^3$$

Hence $my^4 - idy^3 = (a - m)\,y^4 + (b - d)\,y^3$

i.e. $q\,(y) = \frac{1}{2}\,(ay^4 + (1 + i)\,by^3\,)$

Thus $f(x,y) = (\frac{1}{2}(ay^4 + (1 + i)\,by^3) + x, y) \circ (-x,\ iy) \circ$

$$(-\tfrac{1}{2}\,(ay^4 + (1 + i)\,by^3) + x,\ y)$$

and is conjugate to the linear map

$$(x,y) \longmapsto (-x,\ iy)$$

To complete our proof of theorem 1 we now consider the more general case:

Theorem 3: Let f be a polynomial map which is a product of irreducible maps. If f has finite order n, then f is conjugate to a linear map.

Proof: We shall show that such a map is conjugate to a map of lower degree (which still has finite order n) and the theorem then follows by induction - with theorem 2 as the anchoring step.

We can write f as a product:

$$g_r \circ M \circ g_{r-1} \circ \ldots\ldots\ldots \circ g_1$$

with M a linear map (of finite order n) and the maps $g_r, \ldots\ldots, g_1$ irreducible maps with linear parts the identity.

Note that the degree of f is the product of the degrees of g_i's.

Then arguing as in the proof of theorem 2 we can assume that f has been suitably conjugated so that M is of the form: $(x,y) \longmapsto (w_1\ x,\ w_2\ y)$ with w_1, w_2 n-th roots of unity.

As before we take our standard form for each g_i and so we suppose that each factor g_i is of the form:

$$(x,y) \longmapsto (t_i(s_i x + t_i y)^{k_i} + \ldots\ldots, \; - s_i(s_i x + t_i y)^{k_i} + \ldots.)$$

Now conjugate f by the map $N(x,y) = (x + \frac{t_1}{s_1}y, \; y)$ with inverse $N^{-1}(x,y) = (x - \frac{t_1}{s_1}y, y)$ to get:

$$f_1(x,y) = N \circ g_r \circ M \circ g_{r-1} \circ \ldots\ldots \circ g_2 \circ (g_1 \circ N^{-1}).$$

(if $s_1 = 0$ then we must have $t_1 \neq 0$ and then if we conjugate by the map $(x,y) \longmapsto (y,x)$ we recover the form we want.)

The map $h_1 = g_1 \circ N^{-1}$ is of the form:

$$(x,y) \longmapsto (t_1 s_1^{k_1} x^{k_1} + \text{lower terms}, \; - s_1^{k_1+1} x^{k_1} + \text{lower terms})$$

By straight forward calculation we have:

<u>Lemma</u>: The composite of the maps

$(x,y) \longmapsto (t(sx + ty)^k + \text{lower terms}, - s(sx + ty)^k + \text{lower terms})$ and

$(x,y) \longmapsto (ax^m + \text{lower terms}, \; bx^m + \text{lower terms})$ is a map:

$(x,y) \longmapsto (tcx^{km} + \text{lower terms}, -scx^{km} + \text{lower terms})$ for some $c \in \mathbb{C}$.

Applying this to the product:

$$g_r \circ M \circ g_{r-1} \circ \ldots\ldots \circ g_2 \circ (g_1 \circ N^{-1})$$

shows that this map is of the form:

$$(x,y) \longmapsto (t_r cx^k + \text{lower terms}, \; - s_r cx^k + \text{lower terms})$$

where k is the degree of f and $c \in \mathbb{C}$.

Then $f_1(x,y)$ is the composition of N with this map and is:

$$(x,y) \longmapsto \left(\left(t_r - \frac{t_1}{s_1} s_r\right) cx^k + \text{lower terms}, - s_r\, cx^k + \text{lower terms}\right)$$

We cannot have $c = 0$ otherwise the degree of f is less than we assumed and so by the lemma 1 used in proof of theorem 2 we have:

$t_r - \dfrac{t_1}{s_1} s_r = 0$ and so we may put $t_r = e\, t_1$ and $s_r = e\, s_1$

for some $e \in \mathbb{C}$.

Then if we put $h_r = N \circ g_r$ we have

$$h_r(x,y) = \left(x + \frac{t_1}{s_1} y, - e^{k_r+1} s_1 (s_1 x + t_1 y)^{k_r} + \text{lower terms}\right)$$

(This follows from the standard form of Chapter 4)

Hence $f_1 = h_r \circ M \circ g_{r-1} \circ \ldots\ldots \circ g_2 \circ h_1$.
Conjugate f_1 by the map h_1 to get

$$f_2 = h_1 \circ h_r \circ M \circ g_{r-1} \circ \ldots\ldots \circ g_2$$

and observe that $h_1 \circ h_r$ has degree less than the product $k_1 k_r$ of the degrees of g_1 and g_r and so f_2 has degree less than f_1 and hence less than that of f. So we may assume by induction that f_2 (which still has finite order n) is conjugate to a linear map. Hence f_1 is also conjugate to a linear map. This complete the proof of theorem 3 and hence of our main theorem 1.

Remark: We can use our theorem to show that some maps can never have finite order.

In particular we have

Corollary: A reducible map whose degree is the product of different primes can never have finite order.

Proof: Since any map of prime order is irreducible the reducible map which is a conjugate of a linear map can never have a degree which is a product of different primes.

Examples: We can now classify the finite order maps for some lower degrees.

Any map of finite order of degree 2, 3, 5 or 7 must be a conjugate of a linear map by a map of the same degree. (See the remark following theorem 2).

A map of finite order of degree 6 or 10 must be irreducible (from the above corollary) and so the same remark applies to this and so such a map is also a conjugate of a linear map by a map of the same degree.

Maps of finite order of degree 4 may either be irreducible (and the remark above can be applied) or the conjugate of a linear map by a map of degree 2.

The situation for degree 8 maps is a little more complicated. Such a map of finite order can either be irreducible (as above) or the conjugate of a finite order degree 2 map by a degree 2 map (and hence the conjugate of a linear map by a reducible degree 4 map). Such a map is the composition of three degree 2 maps. We can use an argument similar to that in the corollary above to show that a map which is written as a product of irreducibles as (degree 4 map) o (degree 2 map) or (degree 2 map) o (degree 4 map) can never have finite order.

Finally the case of degree 9 maps is similar to that of degree 4 with finite order maps either irreducible or the conjugate of linear maps by maps of degree 3.

CONCLUSION

We collect here some of the more important results proved in the earlier chapters.

1) Let $f(x,y) = (p_r + p_{r-1} + \ldots., q_s + q_{s-1} + \ldots\ldots)$

be a non-linear polynomial map with constant Jacobian. If p_r , q_s are the leading terms and are homogeneous polynomials of degrees r,s respectively in x,y then

$$p_r = c\ (q_s)^{r/s} \text{ for some } c \in \mathbb{C}$$

From this we can deduce:

2) To prove the Jacobian conjecture it is enough to prove that for any polynomial map with constant non-zero Jacobian determinant

$$f(x,y) = (p(x,y),\ q(x,y))$$

the degree of p divides the degree of q or vice-versa.

3) Any invertible polynomial map can be written as a product of the form:

$$U_1 \circ L_1 \circ \ldots\ldots \circ U_k \circ L_k \circ M$$

where U_i is an upper triangular map of the form

$$(x,y) \longmapsto (g_i(y) + x, y) \text{ for some polynomial } g_i,$$

L_i is a lower triangular map of the form

$$(x,y) \longmapsto (x, y + h_i(x)) \text{ for some polynomial } h_i.$$

and M is a linear map.

Moreover we may assume that degree $(U_i) \geq 2$ for $i \geq 2$ and that degree $(L_i) \geq 2$ for $i < k$.

The degree of the invertible map is the product of the degrees of the factors.

4) An invertible polynomial map with linear part the identity of degree $\leq r$ which cannot be written as a product of maps of lower degree can be written in the form:

$$(x,y) \longmapsto (t_{r-1}(s_{r-1}x + t_{r-1}y)^r + \ldots + t_1(s_1x + t_1y)^2 + x, - s_{r-1}(s_{r-1}x + t_{r-1}y)^r - \ldots - s_1(s_1x + t_1y)^2 + y) \qquad \text{for}$$

$s_{r-1}, t_{r-1}, \ldots, s_1, t_1 \in \mathbb{C}$ and satisfying

$s_i t_j = s_j t_i$ for each i, j.

5) An invertible polynomial map f with linear part the identity can be written as a product of irreducible maps with linear parts the identity in such a way that the products of the degrees of the factors is the degree of f, in a unique way.

(Then 4) above allows us to write down a standard form for such a map).

6) The irreducible invertible polynomial maps with linear part the identity of degree less than or equal to r form a variety V_r which can be identified with the quotient of the variety $W_r \subset \mathbb{C}^{2r-2}$ by the action of the product of cyclic groups $C_{r+1} \times C_r \times \ldots \times C_3$ where W_r is the affine variety

$$V(\{ s_i t_j - s_j t_i \mid 1 \leq i, j \leq r-1 \})$$

and the group C_i acts on the (i-2)-th factor of

$$\mathbb{C}^{2r-2} = \mathbb{C}^2 \times \mathbb{C}^2 \times \ldots \times \mathbb{C}^2$$

by: $w(s,t) = (ws, wt)$ where w is an i-th root of unity.

7) The invertible polynomial maps of degree less than or equal to r with linear part the identity form a space U_r which can be constructed from products of the above spaces V_i by making appropriate identifications.

8) The invertible polynomial maps of degree less than or equal to r form a variety

$$GL\,(2, \mathbb{C}) \times U_r$$

This variety has dimension $r + 4$.

9) The cyclic group C_2 acts on the space V_r by $f \mapsto f^{-1}$ and this can be identified with the action induced from the actions of C_2 on each factor $\mathbb{C}^2/C_k$ given by multiplication by an k-th root of -1.

The group $GL\,(2, \mathbb{C})$ acts on V_r by conjugation and this action can be identified with the action induced from the action of $GL\,(2, \mathbb{C})$ on $\mathbb{C}^2/_{C_k}$ given by:

$$(s,t) \longmapsto \Delta_A^{-\frac{1}{k}} \; M_A^{tr} \; (s,t)$$

where Δ_A is the determinant of the matrix M_A of

$A \in GL\ (2,\ \mathbb{C})$

10) All invertible polynomial elements of finite order are conjugate to linear maps. From this we may deduce several results like: A reducible map whose degree is the product of distinct primes cannot have finite order

BIBLIOGRAPHY

1. [BCW] H Bass, E H Connell and D Wright, the Jacobian Conjecture: Reduction of degree and formal expansion of the inverse, Bull Amer Math Soc 7(1982) 287 - 330

2. [K] O H Keller, Ganze Cremona-Transformationen, Monats. Math. Physik 47(1939) 299 - 306

3. [J] H W E Jung, Über ganze birational Transformationen der Ebene, J Reine Angew Math 184 (1942), 161 - 174.

4. [V] Van der Kulk, W, On polynomials rings in two variables, Nieuw Arch Wisk. 1(1953), 33 - 41.

5. [M] A Magnus, On polynomial solutions of a differential equation, Math Scand 3 (1955), 255 - 260.

6. [NB] Y Nakai and K Baba, A generalization of Magnus's theorem, Osaka J Math 14 (1977), 403 - 409.

7. [S] B Segre, Corrispondenze di Mobius e Transformazioni cremonaine intere, Atti della Accademia delle Scienze di Torino, Classe di Scienzo Fisiche, Mat e Natural, 91 (1956 - 57), 3 - 19.

8. [AM] S S Abhyankar and T T Moh, Embeddings of the line in the plane, J Reine Angew Math 276 (1975), 149 - 166.

9. [Vi] A G Vitushkin, On Polynomial Transformations of $\mathbb{C}^n$ (Tokyo, 1973), Tokyo University Press, Tokyo, 1975, 415 - 417.

10. [Mo] T T Moh, On the Jacobian Conjecture and the configuration of roots, J Reine Angew Math 340 (1983) 140 - 212

11. [Me] G H Meisters, Jacobian problems in differential equations and algebraic geometry, Rocky Mountain J Math 12 (1982) 679 - 706